THE GENERAL CHINESE REMAINDER THEOREM

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1. Introduction. The Chinese remainder theorem, as one knows, is one of the most useful tools of elementary number theory. It presents a simple method of determining and representing the solution of a system of simultaneous congruences,

$$x \equiv a_i \pmod{m_i} \qquad (i = 1, 2, \cdots, k), \tag{1}$$

provided the moduls m_i are relatively prime in pairs.

I have been unable to find mentioned anywhere in the mathematical literature the fact that it is possible to formulate a general Chinese remainder theorem, which includes the ordinary, but is valid without any restrictions on the moduls. This is the main result, Theorem 1, in the present paper. It has been expressed for rational integers only, but anyone familiar with the concepts of newer algebra will see how it may be formulated for ideals in quite general rings, so that this need not be elaborated. There have been added, however, some remarks on Abelian groups which appear quite naturally in this connection.

2. The remainder theorem. Let the system of congruences (1) be given where it is not assumed that the moduls are necessarily relatively prime. We recall that in this case the congruences do not always have a solution: the necessary and sufficient condition that they be solvable is that for all i and j,

$$a_i \equiv a_j \pmod{d_{ij}}, \tag{2}$$

where we have put

$$d_{ij} = m_i \wedge m_j \tag{3}$$

for the greatest common divisor of the two moduls m_i and m_j. When the conditions (2) are fulfilled the solution of (1) is uniquely determined for the least common multiple

$$M = m_1 \vee \cdots \vee m_k \tag{4}$$

as modul.

To formulate our theorem we need some further terminology. For each i we shall write

$$A_i = m_1 \vee \cdots \vee m_{i-1} \vee m_{i+1} \vee \cdots \vee m_k$$

and

$$D_i = m_i \wedge A_i = d_{i,1} \vee \cdots \vee d_{i,i-1} \vee d_{i,i+1} \vee \cdots \vee d_{ik}.$$

The quotient of these two numbers is the integer

$$B_i = A_i / D_i. \tag{5}$$

This expression (5) for B_i may be given various other forms by means of the well known identity

$$a \cdot b = (a \vee b) \cdot (a \wedge b). \tag{6}$$

First we have

$$B_i = \frac{A_i}{m_i \wedge A_i} = \frac{m_i \vee A_i}{m_i} = \frac{M}{m_i}. \tag{7}$$

This may be written

$$B_i = \frac{m_1 \vee m_i}{m_i} \vee \frac{m_2 \vee m_i}{m_i} \vee \cdots,$$

which again by (6) and (3) reduces to

$$B_i = \frac{m_1}{d_{1i}} \vee \frac{m_2}{d_{2i}} \vee \cdots. \tag{8}$$

After these preparations we state the main theorem:

THEOREM 1. *The solution of the simultaneous congruences* (1) *can be presented in the form*

$$x \equiv a_1 c_1 \frac{M}{m_1} + \cdots + a_k c_k \frac{M}{m_k} \pmod{M}, \tag{9}$$

where the c_i form a set of integers satisfying the condition

$$c_1 \frac{M}{m_1} + \cdots + c_k \frac{M}{m_k} = 1. \tag{10}$$

Proof. We must first show that the numbers (7) have no common factor so that the indeterminate equation (10) has a solution set $\{c_i\}$. It suffices to show that for each prime p at least one of the numbers B_i is not divisible by p. Let p^{α_i} be the highest power of p dividing m_i and suppose for instance that α_1 is the greatest among these exponents. Then M is divisible exactly by p^{α_1} and B_1 is not divisible by p.

To prove that the expression x in (9) actually is a solution of (1) let us examine it (mod m_j). When the congruence (2) is multiplied by B_i we obtain

$$B_i a_i \equiv B_i a_j \pmod{d_{ij} \cdot B_i}.$$

But according to (8) the number B_i is divisible by m_j/d_{ij} so that we conclude that

$$B_i a_i \equiv B_i a_j \pmod{m_j}.$$

For each i we multiply this congruence by c_i. When the resulting congruences are added one finds

$$x \equiv a_j(B_1c_1 + \cdots + B_kc_k) \pmod{m_j}$$

or, according to (10),

$$x \equiv a_j \pmod{m_j},$$

as desired.

It may be noticed that in the general solution (9) the multipliers for the residues a_i are independent of the particular set of remainders involved in the system (1).

3. Symmetric functions. It is convenient to introduce operational symbols $\vee$ and $\wedge$ for taking the l.c.m. and g.c.d. of a set of numbers just as we use the sum sign $\sum$ and the product sign Π. Thus we write

$$\bigvee n_i = n_1 \vee \cdots \vee n_s$$
$$\bigwedge n_i = n_1 \wedge \cdots \wedge n_s$$

for the g.c.d. and l.c.m. of the numbers n_i.

As before let

$$m_1, m_2, \cdots, m_k \tag{11}$$

denote k arbitrary positive integers. From these we form a system of *symmetric functions* by means of g.c.d. and l.c.m. operations, namely

$$M_r = \bigvee m_{i_1} \wedge m_{i_2} \wedge \cdots \wedge m_{i_r}, \qquad (r = 1, 2, \cdots, k) \tag{12}$$
$$N_r = \bigwedge m_{i_1} \vee m_{i_2} \vee \cdots \vee m_{i_r},$$

where, for each r, the choice of indices,

$$i_1, i_2, \cdots, i_r,$$

runs through all possible $C_{k,r}$ combinations of the numbers from 1 to k: in particular

$$M_1 = m_1 \vee \cdots \vee m_r,$$
$$N_1 = m_1 \wedge \cdots \wedge m_r.$$

From the distributive law for the g.c.d. and l.c.m. (see f. inst. Ore [3], chap. 5-4) one deduces from (12) that

$$M_r = N_{k-r+1}, \tag{13}$$

and it follows also fairly directly from the definitions (12) that

$$M_1 \geqq M_2 \geqq \cdots \geqq M_k, \tag{14}$$
$$N_k \geqq N_{k-1} \geqq \cdots \geqq N_1,$$

where each term divides the preceding.

When k is an odd number one obtains from (13) the self-dual identity

$$M_{(k+1)/2} = N_{(k+1)/2},$$

as has recently been pointed out by Mitrinovitch [1, 2].

These results can also be obtained quite simply by considering the exponents of the various powers of a prime p which enter into the series of numbers (12). As before let m_i be divisible exactly by p^{α_i}; since our expressions are symmetric in the m_i there is no limitation in assuming the notation such that

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_k.$$

Thus M_i is divisible exactly by p^{α_i}, that is by p to the i-th largest exponent to which it occurs, and N_i is divisible by exactly $p^{\alpha_i - k+1}$, that is, p to the $(i-k+1)$-st highest exponent α. Since this holds for every prime p the relations (13) and (14) follow.

An easy consequence of this point of view is the formula

$$m_1 \cdot m_2 \cdots m_k = M_1 \cdot M_2 \cdots M_k, \tag{15}$$

which is a generalization of the simple relation (6). Both sides in (15) contain p to a power with the exponent

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

4. Abelian groups. The k-tuples

$$(x_i) = (x_1, x_2, \cdots, x_k) \tag{16}$$

with integral x_i form an Abelian group under addition. When we suppose further that each x_i is reduced modul m_i, we obtain the general finite Abelian group A of the type

$$\mu = [m_1, m_2, \cdots, m_k], \tag{17}$$

that is, the group with k basis elements of the respective orders m_i.

This representation of the Abelian group by its type is not unique since there may be several types giving the same group. To obtain a unique type representation one can split each m_i into its prime power factors p^{α_i} and represent the type of A by means of the *invariants*

$$\mu_i = [\cdots, p^{\alpha_i}, \cdots]. \tag{18}$$

A second way of obtaining a unique type representation is by means of the *elementary divisors* where one imposes the restriction on the type that each exponent shall be a divisor of the preceding. To construct this elementary divisor type from the invariant type (18) one can proceed as follows. For each prime p one takes the highest invariant p^{α_i} and then one forms the product of these p^{α_i} over all the primes p in (18). Next one takes the second highest invariants and their product, and so on. These products are evidently the symmetric functions $M_1, M_2, \cdots$ so that we can state:

THEOREM 2. *The elementary divisor type of an Abelian group of type* (17) *is*

$$\mu_e = [M_1, M_2, \cdots, M_k],$$

where the numbers M_j are the symmetric functions of the numbers m_i.

From this point of view the formula (15) expresses only the fact that the order of the group is independent of the type representation.

Let us now consider what we may call the *residue k-tuples*

$$(a_i) = (a_1, a_2, \cdots, a_k). \tag{19}$$

These are the elements in A for which the corresponding congruences (1) are solvable, that is, the sets of numbers for which the congruence conditions (2) are fulfilled. It is evident that these residue k-tuples (19) form a subgroup, the *residue group* R of A. But since the congruences (1) are solvable in this case, each a_i may be replaced by the same x so that the residue group R is simply the subgroup of A consisting of all k-tuples

$$(x) = (x, x, \cdots, x)$$

in which all components may be taken to be the same. We notice that the order of R is M_1.

Finally one may say that two k-tuples (x_i) and (y_i) in A are *residue equivalent* if their difference is a residue k-tuple (19). The residue equivalence classes also form a group, the *residue difference group* D isomorphic to the difference group $A-R$ (quotient group A/R). The order of this group is

$$\frac{m_1 \cdots m_k}{M_1} = M_2 \cdots M_k,$$

and we leave it to the reader to verify that its elementary divisors are actually

$$M_2, M_3, \cdots, M_k.$$

Instead of taking the additive group A one could have considered the multiplicative group A' consisting of those k-tuples in (16) in which each component x_i is relatively prime to its corresponding modul. The type of this group is

$$\mu' = [\phi(m_1), \cdots, \phi(m_k)]$$

where, as usual, ϕ denotes Euler's function. Here our symmetric functions $M_j(m_i)$ are replaced by $M_j(\phi(m_i))$. The order of the group may be written in several forms:

$$\begin{aligned} \phi(m_1) \cdots \phi(m_k) &= M_1(\phi(m_i)) \cdots M_k(\phi(m_i)) \\ &= \phi(M_1(m_i)) \cdots \phi(M_k(m_i)). \end{aligned} \tag{20}$$

These identities (20) are extensions of the simple rule

$$\phi(a) \cdot \phi(b) = \phi(a \vee b) \cdot \phi(a \wedge b) = (\phi(a) \vee \phi(b)) \cdot (\phi(a) \wedge \phi(b)).$$

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THE SUMS OF THE DIHEDRAL AND TRIHEDRAL ANGLES IN A TETRAHEDRON*

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There is no theorem on the sum S of the dihedral angles in a tetrahedron analogous to the theorem that the sum of the angles in a triangle is π radians. Furthermore, a little experimentation shows that the sum is not a constant. However, bounds are known. It is well known that the sum of the three dihedral angles around any vertex is between π and 3π, from which it follows immediately that S is between 2π and 6π. These are the best bounds the writer has been able to find in the literature.

The principal result of this note is to show that S is between 2π and 3π, and that these bounds cannot be improved. Since T, the sum of the trihedral angles, is given by $T=2S-4\pi$, this also gives best bounds on T. (One would expect these results to be well known and the writer would appreciate information on this point. In any event, they do not appear to be readily accessible and the elementary proofs given here may be of value.)

We first prove a lemma about spherical triangles.

LEMMA. *If x is a point interior to a spherical triangle abc, then $ax+bx+cx \leqq ab+bc+ac$.*

Proof. Extend ax, bx, cx, intersecting bc, ac, and ab, respectively in p, q, r. Then $cx+ax-ar \leqq cx+xr \leqq br+bc$. Hence $cx+ax \leqq ar+br+bc=ab+bc$. Similarly, $ax+bx \leqq bc+ac$, and $cx+bx \leqq ab+ac$. Adding and dividing by 2, $ax+bx+cx \leqq ab+bc+ac$.

The proof is clearly valid for plane triangles as well.

Proceeding to consider the theorem, given any tetrahedron, let us take an interior point O and drop perpendiculars r_1, r_2, r_3, r_4 to the faces. Now the six dihedral angles of the tetrahedron are supplementary, respectively, to the six

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angles formed by r_1, r_2, r_3, r_4. That is, denoting by R the sum of the angles formed by the r_i, we obtain $S=6\pi-R$.

We are now concerned with finding bounds for R. If we draw a unit sphere about O, r_1, r_2, r_3, r_4 intersect this sphere in points p_1, p_2, p_3, p_4 and R is the sum of the six spherical distances p_ip_j. The fact that O is interior to the tetrahedron is equivalent to the fact that the points p_i are not contained in any hemisphere, or as we may say, are global.

THEOREM. *If R is the sum of the six distances between points of a global quadruple, $3\pi \leqq R$.*

Proof. Denoting the points p_1, p_2, p_3, p_4 and their antipodal points p_1', p_2', p_3', p_4', we wish to show first that p_1' is interior to the triangle $p_2p_3p_4$. (The argument given here was formulated by Professor L. M. Blumenthal.) Since the quadruple is global, p_1' and p_4 are on the same side of the great circle p_2p_3. Similarly p_1' and p_2' are on the same side of p_3p_4, and p_1' and p_3 are on the same side of p_2p_4. Hence p_1' is in the triangle $p_2p_3p_4$.

Now $p_1p_2+p_2p_1'=p_1p_3+p_3p_1'=p_1p_4+p_4p_1'=\pi$. But by the lemma, $p_2p_1'+p_3p_1'+p_4p_1' \leqq p_2p_3+p_3p_4+p_2p_4$. Hence

$$p_1p_2 + p_2p_1' + p_1p_3 + p_3p_1' + p_1p_4 + p_4p_1' = 3\pi \leqq R = p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_3p_4 + p_2p_4.$$

COROLLARY. $2\pi \leqq S \leqq 3\pi$.

Proof. Since $S=6\pi-R$, the preceding theorem gives $S \leqq 3\pi$. While it is known that $S \geqq 2\pi$, as remarked in the first paragraph, it is interesting to give a proof in the spirit of this note. We have to prove that $R \leqq 4\pi$. Since the perimeter of any triangle (on the unit sphere) is at most 2π, adding the perimeters of the four triangles in any quadruple gives $2R \leqq 8\pi$, and hence $S \geqq 2\pi$.

It is to be remarked that these bounds on S are the best possible. Examples to show this are not difficult to find, but a simple argument, suggested by the referee, is convincing. We simply observe that we can find a tetrahedron arbitrarily close to either of the two limiting forms of zero volume, one of which is a plane quadrangle with its diagonals, the angle sum being 2π, the other of which is a triangle with its three vertices joined to an interior point, with angle sum 3π.

COROLLARY. The best bounds on T are $0 \leqq T \leqq 2\pi$.

This follows from the formula $T=2S-4\pi$.

These methods can be used to characterize tetrahedra whose dihedral angles are less than $\pi/3$ or greater than $2\pi/3$, that is, which have no dihedral angles lying between $\pi/3$ and $2\pi/3$. An examination of cases shows that such a tetrahedron either has two obtuse angles, these being opposite angles, or three obtuse angles, these emanating from one vertex.

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

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1. Introduction. A recent paper in this MONTHLY [1] has attempted to base a proof of the Fundamental Theorem of Algebra on Brouwer's Fixed Point Theorem. Since, as is pointed out in [2], this paper contains a basic error, a correct proof of the Fundamental Theorem of Algebra which is based on the Brouwer theorem may be of interest.

In this paper we give a brief exposition of some well known theorems about continuous functions, including the Brouwer theorem for the plane and the Fundamental Theorem of Algebra. The development presented here presupposes a knowledge of only the most elementary properties of continuous functions and complex numbers. A similar expository article has been written by A. W. Tucker [3]. Both [3] and [4] are highly recommended to those who are interested in elementary treatments of some important topological theorems.

2. Existence of continuous logarithms. We let P be the plane of complex numbers, P_0 be the set of all non-zero complex numbers, and S be the set of all $z \epsilon P$ for which $|z| = 1$. If $z \epsilon S$, then we let $A(z)$ be the set of all real numbers θ for which $z = e^{i\theta}$; (*i.e.* $A(z)$ is the set of all arguments of z). By a *disk* we mean the union of a circle and its interior. The word *mapping* is used to denote a continuous function. If f is a mapping of a set X into S, then *f has a continuous logarithm on X* if there exists a real-valued mapping ϕ of X such that $f(z) = e^{i\phi(z)}$ for all $z \epsilon X$. A systematic treatment of logarithms is given by Eilenberg in [4].

We shall assume and make use of the following two elementary properties of complex numbers:

(i) If w_1 and w_2 are points of S, $|w_1 - w_2| < 2$ and $\theta_1 \epsilon A(w_1)$, then there exists a unique $\theta_2 \epsilon A(w_2)$ such that $|\theta_1 - \theta_2| < \pi$.

(ii) If $\theta_1 \epsilon A(w_1)$, $\theta_2 \epsilon A(w_2)$ and $|\theta_1 - \theta_2| < \pi$, then $|\theta_1 - \theta_2| \leqq \pi |w_1 - w_2|$.

Property (ii) merely states that if an arc is contained in a semi-circle, then the length of the arc is less than π times the length of the chord joining the end points of the arc.

THEOREM 1. *If f is a mapping of a disk D into S, then f has a continuous logarithm.*

Proof. The mapping f is uniformly continuous on D, and hence there exists $\delta > 0$ such that if z_1 and z_2 are in D and $|z_1 - z_2| < \delta$ then $|f(z_1) - f(z_2)| < 1/3$. Let q be the center of D and r be the radius of D. We choose a positive integer n such that $r/n < \delta$, and then define $D_k = \{z \mid |z - q| \leqq kr/n\}$ for each integer k, $0 \leqq k \leqq n$. We now define a function ϕ on D by defining ϕ successively on $D_0, \cdots, D_n$. We first define ϕ on D_0 by letting $\phi(q)$ be the $\theta \epsilon A(f(q))$ for which $0 \leqq \theta < 2\pi$. If ϕ has been defined on D_k and $z \epsilon D_{k+1}$, let z' be the point of D_k which is nearest z. Since $|z - z'| < \delta$, $|f(z) - f(z')| < 1/3$; hence by (i) we may define $\phi(z)$ to be the unique member of $A(f(z))$ which differs from $\phi(z')$ by less than π.

Now let S_k be the statement: If z_1 and z_2 are in D_k and $|z_1-z_2|<\delta$, then $|\phi(z_1)-\phi(z_2)|<\pi$. We shall prove successively that $S_0, \cdots, S_n$ are true. S_0 is obviously true since D_0 contains only one point. Now let us assume that we have proved S_k true. Let z_1 and z_2 belong to D_{k+1} and satisfy $|z_1-z_2|<\delta$. We define z_1' and z_2' to be the points of D_k which are the nearest to z_1 and z_2 respectively. It follows from the way in which ϕ is defined on D_{k+1} that $|\phi(z_1)-\phi(z_1')|<\pi$ and $|\phi(z_2)-\phi(z_2')|<\pi$. It is easy to see that $|z_1'-z_2'|<\delta$, and since S_k is true we obtain $|\phi(z_1')-\phi(z_2')|<\pi$. Making use of (ii) and the way in which δ was chosen, we obtain

$$\begin{aligned}|\phi(z_1)-\phi(z_2)| &\leqq |\phi(z_1)-\phi(z_1')|+|\phi(z_1')-\phi(z_2')|+|\phi(z_2'-\phi(z_2)| \\ &\leqq \pi|f(z_1)-f(z_1')|+\pi|f(z_1')-f(z_2')|+\pi|f(z_2')-f(z_2)| \\ &< \pi/3+\pi/3+\pi/3=\pi.\end{aligned}$$

This proves that S_{k+1} is true. Thus we see that S_j is true for $0\leqq j\leqq n$.

From S_n and (ii), we see that if z_1 and z_2 are in D and $|z_1-z_2|<\delta$ then $|\phi(z_1)-\phi(z_2)|\leqq\pi|f(z_1)-f(z_2)|$. The continuity of ϕ now follows easily from the continuity of f. Since $\phi(z)\epsilon A(f(z))$ for each $z\epsilon D$, we have shown that f has a continuous logarithm on D.

Let $I(z)=z$ for each $z\epsilon S$. In contrast to the above theorem we now prove that there are mappings of S into S which do not have continuous logarithms on S.

THEOREM 2. *The identity function I on S does not have a continuous logarithm on S.*

Proof. Suppose that I has a continuous logarithm on S, and that ϕ is a continuous function such that $I(z)=z=e^{i\phi(z)}$ for all $z\epsilon S$. We can define a continuous function g on the closed interval $[0, 2\pi]$ by letting $g(\theta)=\phi(e^{i\theta})-\theta$. For each θ, $g(\theta)$ is an integral multiple of 2π. Since g is continuous, this implies that g is a constant function. This is impossible, however, since $g(0)=\phi(1)$ and $g(2\pi)=\phi(1)-2\pi$.

3. The Brouwer Fixed Point Theorem. In order to prove Brouwer's theorem, we shall need the following theorem which is itself an important theorem of topology.

THEOREM 3. *If D is the disk formed by the union of S with its interior, then there does not exist a mapping f of D into S such that $f(z)=z$ for all $z\epsilon S$.*

Proof. If such a mapping f did exist, then by Theorem 1 there is a mapping ϕ on D such that $f(z)=e^{i\phi(z)}$ for all $z\epsilon D$. By restricting the domain of ϕ, we see that f has a continuous logarithm on S. However, since f is the identity mapping on S, this contradicts Theorem 2.

THEOREM 4 (Brouwer Fixed Point Theorem). *If g is a mapping of a disk D into D, then there exists a point $z\epsilon D$ such that $g(z)=z$.*

Proof. There is no loss in generality in assuming that D is the disk which is the union of S and the interior of S, since all other cases can be reduced to this one by a change of coordinates.

Suppose $g(z) \neq z$ for each $z \epsilon D$. Then, for each $z \epsilon D$, there is a unique point $f(z)$ on S such that z is on the segment joining $f(z)$ to $g(z)$. The function f so defined is continuous, is a mapping of D into S, and $f(z) = z$ for all $z \epsilon S$. We have a contradiction, since Theorem 3 states that such a function does not exist.

4. The existence of continuous *n*th roots.

THEOREM 5. *If f is a mapping of a disk D into P_0 and n is a positive integer, then there exist n distinct mappings $h_1, \cdots, h_n$ which have the property that $[h_k(z)]^n = f(z)$ for all $z \epsilon D$.*

Proof. We can write $f(z) = r(z)g(z)$ where $r(z) = |f(z)|$ and $g(z) = f(z)/|f(z)|$. Then r is a mapping of D into the set of positive real numbers and g is a mapping of D into S. By Theorem 1, there exists a mapping ϕ such that $g(z) = e^{i\phi(z)}$. We get the desired n-th roots of f by defining

$$h_k(z) = \sqrt[n]{r(z)}\, e^{(\phi(z)+2k\pi)i/n}.$$

5. Fundamental Theorem of Algebra.

THEOREM 6 (Fundamental Theorem of Algebra). *If p is a polynomial of degree $n > 0$, then there exists $z \epsilon P$ such that $p(z) = 0$.*

Proof. Let us assume that the coefficient of z^n in $p(z)$ is $1/2$. We may clearly do this without loss of generality. We now assume that $p(z) \neq 0$ for all $z \epsilon P$ and prove our theorem by establishing a contradiction.

Since $\lim_{z \to \infty} p(z)/z^n = 1/2$, there exists $r > 0$ such that if $|z| \geqq r$, then $p(z)$ and z^n have arguments which differ by less than $\pi/3$ and also $|p(z)| < |z|^n$. Now choose $R > r$ such that $\sqrt[n]{|p(z)|} + |z| < R$ for $|z| \leqq r$. We define D to be the disk with center at 0 and radius R. Since p maps D into P_0, p has a continuous n-th root f on D. We choose f so that $f(r)$ and r have arguments which differ by less than $\pi/3n$. It follows from an easy continuity discussion that $f(z)$ and z have arguments which differ by less than $\pi/3n$ for all z such that $|z| \geqq r$. (Otherwise, for some z such that $|z| \geqq r$, $f(z)$ and z have arguments differing by exactly $\pi/3n$; this implies that $p(z)$ and z^n have arguments differing by exactly $\pi/3$, which is impossible.)

Since $|p(z)| < |z|^n$ for $|z| \geqq r$, we obtain $|f(z)| < |z|$ for $|z| \geqq r$. Thus, if $|z| \geqq r$, z and $f(z)$ both lie in a circular sector of radius $|z|$ and angle $\pi/3n$. It is therefore easy to see that $|z - f(z)| \leqq |z|$ for $|z| \geqq r$. On the other hand, if $|z| \leqq r$, then $|z - f(z)| \leqq |z| + |f(z)| < R$. If follows that if we define $g(z) = z - f(z)$ for $z \epsilon D$, then g is a mapping of D into D. We may now use the Brouwer theorem to find z_0 such that $g(z_0) = z_0$. It follows that $f(z_0) = 0$ and hence $p(z_0) = 0$. This is a contradiction.

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THE UNSATISFACTORY STORY OF CURVATURE

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1. The Preliminaries. The fact that plane loci are sometimes straight and sometimes are not would seem to be about as obvious as any property of such figures; it is therefore very surprising that this was not placed in the foreground by early writers who dealt with the matter. Euclid defines a straight line as one which lies evenly with the points on itself, which meant that the end points would completely cover those which lay in between or completely hide the latter points, and countless variations of this idea were evolved by the early writers of geometry, but the dynamic idea that a straight line was generated by a point which always moved in the same direction was not, as far as I know, stressed by anyone. Of course whoever admits direction as a primary idea of geometry is involving himself in a sea of trouble, and this may be the reason for its avoidance, but the total neglect of anything so fundamental is certainly striking.

The Greek writers were familiar with all sorts of specific curved loci, witness Proclus' famous summary of the history of geometry, and he says that Aristotle recognized three kinds of loci "*C'est aussi la raison pour laquelle il-y-a trois mouvements, l'une en direction de la ligne droite, l'autre circulaire, et le troisième mixte*" [1]. Proclus himself expands this idea in [1] p. 234, "*Tandis que d'autres ont affectuès la section au moyen des quadratices de Hippias et de Nicomède lesquels avaient aussi fait usage de lignes mixtes quadratices.*" The Greeks were familiar with curved loci, but singularly slow in pointing out the distinguishing characteristics.

The fundamental idea lay very close at hand. Some curves were straight, others were curved. Whatever curvature might be, a circle was everywhere equally curved. The greater the curvature, the less the radius, so it would be natural to take as the curvature a quantity inversely porportional to the radius of the circle, and for any curve, the curvature of the circle lying nearest to it. All this is so simple and natural, but it was strangely slow in coming to be realized. I am sure that a complete account can be found; it certainly is not in [2]. I write in the hope that someone will succeed where I have failed to find it.

The question of curvature is intimately connected with that of the centre of curvature, and Apollonius is perfectly aware that from certain points but one normal can be drawn to a conic, and the careful discussion of this question was earnestly pursued, but the writer did not go on to other loci, and it is curious that neither he nor Euclid in discussing optics took the quite obvious steps in this connection. On the contrary, the attention was diverted to the analogous problem of horn angles. A horn or cornicular angle is roughly the figure formed by the circumference of a circle and its tangent; a good discussion by Heath is found in [3]. We learn by Euclid III, 16 that the angle of the semicircle is greater and the remaining angle is less than any acute rectilineal angle. As long as a secant cuts a circumference twice, the angle which it makes with the curve is less than the horn angle, but the latter, though it can be indefinitely increased, by decreasing the radius of the circle, is nevertheless smaller than a right angle. This is very much contrary to the notion that the two have a definite ratio. Vieta in [4] takes up the question of whether a circumference and its tangent really can be said to make an angle, and points out that in any case this must be a different sort of object from other figures which cannot pass from greater to less without passing through equality. Wallis in [5] insists, in a withering attack on Clavius, that a horn angle is not an angle in Euclid's sense as it is not an inclination between two loci. A new word is needed "*Cui respondeo, mihi cum Clavio hactem convenire (et convenisse semper). Quod quem ille vocat Angulum Contactus, nil aliud est quod ego voco Gradum Curvitatis. Sunt utque curvitatum gradum semper proportionales longitudinibus Diametrorum Chordarium arcuum similibus.*"

I will not go further into the complicated question of horn angles but return to the larger question of definition of curvature. The first writer to give a hint of the definition of curvature was the fourteenth century writer Nicolas Oresme, whose work was called to my attention by Carl Boyer. We find Oresme saying in [6] "*Nunc restat de Curvitate dicendum.*" He assumes the existence of something which he calls *Curvitas*, and if we have two curves touching the same line at the same point, and on the same side, the smaller curve will have the greater curvature. He further states that the curvature of a circle is "*Uniformis*" and on p. 219 "*Sit circulus major cuius semidyameter sit AB et circulus minor cuius semidyameter sit AC, tunc si sit semidyameter AB duplo ad semidyanmeter AC, curvitas minoris circuli erit duplo intensior curvitate majoris, et ita proportionibus et curvitatibus.*" We could not have a clearer proof that Oresme conceived the curvature of a circle as inversely proportional to the radius; how did he find this out?

We apparently have to wait nearly three hundred years before finding anything further on the subject of curvature when we turn to the work of Kepler [7]. He undertakes to find the image of a certain brilliant point, the generalized problem of Al Hazen. He shows the standard approach but adds that instead of dealing with the curve itself it would be wiser to deal with its circle of curvature "*At verior ratio jubet invenire circulum, qui continett rationem curvitatis.*" The ra-

tio he connects with a certain circle, the circle of curvature. His editor, Frischauf in a note on p. 403 to [7] adds "*Primum hic occurit circulus quem dicunt osculatorem, quem posteriores mathematici tum add lineas tum ad superficies curvas maximo commodos adhibebunt, cujus inventio Leibnitio huc usque tributa ets.*" We shall presently see that it was certainly a mistake to associate Leibniz with the invention of the circle of curvature, but we have a glimpse of the fact that the problem of the rightful ascription was not too simple.

A great step in advance is due to Huygens. I mention especially his *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demon strationes geometricae* of 1673 [8]. Kepler starts curiously with the involutes of curves. We take a curve all of whose tangents are on one side. To this is attached a flexible string which is pulled taut and then unwound. The curve from which the string springs is called the evolute. He assumes that the string will always remain tangent to this evolute; the locus of a point fixed in the string will be the involute. He begins with a careful proof that the string will always be normal to the involute. Then comes the curious theorem that two curves with tangents on one side which have a common point cannot have the same set of normals. After this we see that if all tangents to one curve are normals to another they will be an involute and evolute as defined above. He next passes to certain specific curves, notably the cycloid, and in theorem XI he undertakes the problem of finding the radius of curvature of any given curve, defined as the distance up the normal from the foot to the point of contact with the evolute. This is defined as the limit of the intersection with an infinitely near normal. In our notation this would involve Δx and $\Delta[y(dy/dx)]$. He says in regard to this "*Illas vero dari in omnibus curvis geometricis.*" He indicates the method of doing this in general, but his ignorance of the calculus prevented him from attaining a satisfactory result in every case.

2. **Sir Isaac Newton.** The first writer to handle the question of curvature of a plane curve in what we should call today a thoroughly satisfactory manner was Sir Isaac Newton, no less. Huygens wrote the *Horologium oscillatorium* in 1673. Newton started thinking about the Calculus in around 1665 but published nothing on the subject till his letter to Collins of 1669; his ideas were well developed in 1671. The first systematic account appeared in [9] with the date of 1736. There is a long appendix due to Colson himself, but the bulk of the work purports to be a direct translation of Newton's own Latin. I shall return later to the possible relation of this to the *Horologium* of Huygens.

I begin by quoting Newton's own words, pp. 59–61 of [9]:

"The same Circle has everywhere the same curvature, and in different Circles it is reciprocally proportional to their Diameters.

"If a Circle touches any Curve on its concave side, in any given point, and if it be of such magnitude that no other tangent Circle be inscribed in the angle of contact of that Point, that Circle will be of the same Curvature as the Curve is of, in the Point of Contact.

"Therefore the Centre of Curvature to any Point of the Curve is the Centre of the Circle equally curved, and thus the Radius or Semi-diameter of Curvature is Part of the Perpendicular to the Curve which is terminated at the Centre.

"And the Proportion of Curvature at different Points will be known from the Proportion of Curvature of aequi-curve Circles, or from the reciprocal Proportion of the Radii of Curvature."

A further explanation comes presently:

"But there are several Symptoms or Properties of this Point C which may be used in its Determination:

1) That it is the Concourse of Perpendiculars to that Arc on each side at an infinitely little distance from DC.
2) If DC be conceived to move while it insists perpendicularly to the Curve, that point of it C (if you accept the motion of approaching to, or receding from the point of insistance C) will be the least moved, but will be its Centre of Motion.
3) If a Circle be described with the Centre C, and the Distance DC, no other Circle can be described that can lie between it and the Circle of Contact.
4) Lastly if the Centre H or h of any other touching Circle approaches by degrees to C the Centre of this, till at last it coincides with it, then any of the points in which the Circle shall cut the Curve, will coincide with the Point of Contact at D.
5) And each of these Properties may supply the means of solving this Problem in different ways. But we shall make choice of the first, as being the most simple."

What all this amounts to is the following. Newton assumes, as does Huygens, that if a point not an inflection is fixed on a curve, and a second point approaches it from either side, the intersection of the normals approaches a definite limiting position. This is the centre of curvature, the centre of a circle having the same curvature, and no other tangent circle can lie between this and the curve. All of these properties can be proved if one of them is assumed. He seeks the intersection of nearby normals, and if we take x as an independent variable so that $\dot{x}=1$ and $z=\dot{y}/\dot{x}$ he proves very simply

$$DC = \frac{(1+z^2)^{3/2}}{\dot{z}}.$$

This is very satisfactory and may be said to close the question of the discovery of curvature, but there remains a troublesome question of priority. Cantor in his *Geschichte der Mathematik*, Vol. 3, p. 17, raises the question whether between 1671 and 1736 Newton did not see Huygens' work of 1673 and find therein an excellent opportunity to make use of his own vastly superior methods. Newton made reference to flexible strings and pendulums and the preoccupation of both mathematicians with the cycloid troubles Cantor. On the other hand, it should

be noticed that this was a time when a good many geometers were occupied with the cycloid where the determination of the radius was particularly easy. Newton pointed out various methods of finding the radius of curvature, and chooses the best. I cannot feel that we are justified in accusing Newton of plagiarism; it was not in his nature.

There remains the question of Leibniz. In [10] he discusses the angles of mutually tangent curves, and the contact with the "*Circulus osculans*" as if this were an already familiar figure. But the date is late, 1686, and he erroneously says that the osculating circle has four coincident intersections, a mistake promptly pointed out by James Bernoulli.

This represents the limit of my most incomplete knowledge. Where did Oresme get the idea that the curvature of a circle was inversely proportional to the radius? Where did Kepler find the circle of curvature? There are various intersecting questions still to be answered. More power to the persevering man who will answer them.

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A CANONICAL BASIS FOR THE IDEALS OF A POLYNOMIAL DOMAIN

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1. Introduction. This paper is concerned with ideals in the domain of polynomials with integral coefficients. To put the problem in a more general perspective, I begin with compiling some basic algebraic notions and facts. For details the reader may consult Van der Waerden's *Modern Algebra*, Vols. 1 and 2, henceforth quoted as MAI and MAII.*

* Page numbers refer to the English edition, 1949.

An *integral domain* is a commutative ring (MAI, p. 32) with identity element 1, which contains no divisors of zero: $ab=0$ implies either $a=0$ or $b=0$. If R is an integral domain then the polynomials $f(x)=a_0+a_1x+\cdots$ with coefficients in R also form an integral domain (MAI, p. 47), denoted by $R[x]$. If not stated otherwise, R shall always denote an integral domain.

A *unit* of R is an element e which is a divisor of 1: $ee_1=1$ for a suitable $e_1\epsilon R$. We say, a and a_1 are *associates* if $a_1=ea$, e unit of R. An element is *prime* if $p=bc$ implies that either b or c is a unit. In certain integral domains such as the ring of integers, every element can be factored into a product of prime elements, in essentially one way. That is, if a is not a unit and $a=p_1\cdots p_r=q_1\cdots q_s$ where p_μ, q_ν are prime elements, but not units, then $r=s$ and p_ν is associated with q_ν, $\nu=1,\cdots,r$, provided that the factors have been arranged in a suitable order. An integral domain with this property is called a *unique factorization domain*.

The following theorem is essentially due to Gauss (MAI, p. 73):

If the unique factorization theorem holds in R then it also holds in $R[x]$. In particular there is a unique factorization of polynomials with integral coefficients and in any number of variables.

For later purposes we note the following theorem which, in a slightly different form, is known as Gauss' Lemma (MAI, p. 71). It is the usual starting point for the proof of Gauss' theorem.

LEMMA. *If R is a unique factorization domain and $af(x)=g(x)h(x)$ where $g(x)$ is a primitive polynomial then $h(x)=ak(x)$.*

A *primitive polynomial* is one whose coefficients have no common divisor except units. Of course it is understood that the polynomials f, g, h, k belong to $R[x]$.

An *ideal* in R is a non-empty subset $M\subset R$ which has the following properties:

1. $a\epsilon M$, $b\epsilon M$ imply $a-b\epsilon M$.
2. $a\epsilon M$, $c\epsilon R$ imply $ac\epsilon M$.

In particular, if $a_1, a_2, \cdots$ are fixed elements of R, then the set of linear combinations $c_1a_1+\cdots+c_ka_k$, $c_\nu\epsilon R$, $\nu=1,\cdots,k$, $k=1, 2, \cdots$, form an ideal which is denoted by $(a_1, a_2, \cdots,)$. The elements a_ν form a *basis* of the ideal. In the particular case when $k=1$, the ideal (a) consists of the multiples of a and is called a *principal ideal*. There are integral domains called *principal ideal domains* (pr. i.d.), in which every ideal is principal; for example, the ring of integers is a pr.i.d. Also if F is a field, then $F[x]$ is a pr.i.d. (MAI, p. 55).

In a pr.i.d. the prime factorization of an element, if it exists at all, is always unique (MAI, p. 60). The converse is not generally true; *e.g.* if R is the ring of integers then $R[x]$ is a unique factorization domain but not a pr.i.d. Although ideals of $R[x]$ are not necessarily principal, they have at least a finite basis.

That is, if R is a pr.i.d. then every ideal of $R[x]$ has the form

$$M = (f_1(x), \cdots, f_k(x)). \tag{*}$$

This is a consequence of the following more general theorem of Hilbert (MAII, p. 18):

If in R every ideal has a finite basis then also the ideals of $R[x]$ have a finite basis.

Polynomial domains have been studied extensively in the past 100 years from the arithmetical and algebraical point of view. We shall deal here with the enumerative aspect of the problem. Our purpose is to determine, in a purely enumerative manner, all the ideals of $R[x]$ when R is a pr.i.d. At first sight it would seem that Hilbert's basis theorem furnishes a very simple solution of the problem. In fact, all we have to do is to take every finite combination of polynomials and form the ideals (*). But an ideal has usually a great number of representations in the form (*) and the enumeration, although exhaustive, would involve unnecessary repetitions. It is therefore desirable to have a more economical method of enumeration by which every ideal is obtained in one and (possibly) only one way. This we shall achieve by setting up a canonical basis for the ideals of $R[x]$ and deriving certain numerical invariants which characterize the ideals uniquely. To avoid complications, until the last section we shall assume that R is the ring of ordinary integers.

2. If M is a principal ideal $(f(x))$, then we shall call $(f(x))$ or $(-f(x))$ its canonical basis, according as the leading coefficient of $f(x)$ is positive or negative. If $M=(f(x))M_0$ where the leading coefficient of $f(x)$ is positive and M_0 has the canonical basis $(g_1(x), \cdots, g_m(x))$ then the canonical basis for M shall be defined $(f(x)g_1(x), \cdots, f(x)g_m(x))$. Thus it is sufficient to find a canonical basis for every non-principal ideal M whose elements have no common divisor except ± 1. This condition implies that M contains a non-zero constant d. For, let $h(x)=d(a_lx^l+\cdots+a_0)$, $d\neq 0$, $(a_0, \cdots, a_l)=1$, be a polynomial of smallest degree in M and $f(x)$ an arbitrary polynomial of degree $k\geqq l$ in M. By the usual division algorithm we can determine $g(x)$ so that $h_1(x)=da_l^{k-l+1}f(x)-g(x)h(x)$ shall have a degree less than l. But $h_1(x)$ is in M, hence $da_l^{k-l+1}f(x)=g(x)h(x)$ and by Gauss' lemma $f(x)$ is divisible by the primitive polynomial $a_lx^l+\cdots+a_0$. This is true for every $f(x)\epsilon M$, hence $l=0$, $a_0=\pm 1$ and $h(x)=\pm d\epsilon M$.

Since M contains a constant, it contains polynomials of an arbitrary degree k. For every $k\geqq 0$ let a_k be the smallest positive number which is the leading coefficient of a polynomial of degree k in M, and

$$g_k(x) = a_kx^k + \sum_{i=0}^{k-1} a_{ki}x^i, \qquad (k = 0, 1, 2, \cdots),$$

a corresponding sequence of polynomials. We shall call polynomials $g_k(x)\epsilon M$ with this property *minimal*.

By a well-known argument we can show that a_k is a divisor of the leading coefficient of any polynomial of degree k in M. For, let $c_k = qa_k + r$, $0 \leqq r < a_k$, then $f(x) - qg_k(x) = rx^k + \cdots$ is in M; hence by the minimum property of a_k, $r = 0$, $c_k = qa_k$.

It follows that

$$a_k \mid a_{k-1}, \qquad a_{k-1} = q_k a_k \qquad \text{for } k > 0, \tag{1}$$

(2) every $f(x) \epsilon M$ of degree k is already in $M_k = (g_0(x), \cdots, g_k(x))$.

More precisely,

$$f(x) = \sum_{i=0}^{k} b_i g_i(x), \qquad b_k \neq 0 \tag{3}$$

where the b_i are constants. From Hilbert's basis theorem we conclude:
(4) There is an $m > 0$ such that $M \neq M_{m-1}$, $M = M_m$.

Of course the sequence of minimal polynomials $g_k(x)$ is not uniquely determined by M. In fact, every combination

$$g_k'(x) = g_k(x) + \sum_{i=0}^{k-1} b_i g_i(x) \tag{5}$$

is minimal in M. Nevertheless it follows from (2) that M_k is independent of the choice of the sequence $g_k(x)$, and also m in (4) is an invariant. We shall call m the degree of M.

We can always choose the combination (5) such that the coefficients of $g_k'(x) = a_k x^k + \sum_{i=0}^{k-1} a_{ki}' x^i$ satisfy the inequality

$$0 \leqq a_{ki}' < a_i, \qquad (i = 0, \cdots, k-1).$$

Further, $g_k'(x)$ is uniquely determined by this condition. For, suppose that $g_k''(x) = a_k x^k + \sum_{i=0}^{k-1} a_{ki}'' x^i \epsilon M$ would also satisfy $0 \leqq a_{ki}'' < a_i$, $i = 0, \cdots, k-1$. Then $g_k'(x) - g_k''(x) = \sum_{i=0}^{k-1} (a_{ki}' - a_{ki}'') x^i \epsilon M$ with $|a_{ki}' - a_{ki}''| < a_i$ for $i = 0, \cdots, k-1$, which is only possible (by the definition of a_i) if $a_{ki}' = a_{ki}''$ for $i = 0, \cdots, k-1$; hence $g_k'(x) = g_k''(x)$.

We have now a uniquely determined basis in M which, in a sense, could be termed canonical. There is however the following difficulty: suppose we prescribe arbitrary values for a_k and a_{ki} subject to the condition

$$0 \leqq a_{ki} < a_i, \qquad (k = 0, \cdots, m;\ i = 0, \cdots, k-1). \tag{6}$$

Then we may ask the following question: Is it true that the corresponding polynomials $g_k(x)$ form a canonical basis in the above sense for $M = (g_0(x), \cdots, g_m(x))$? The following example shows that this is not necessarily true; in fact the $g_i(x)$ need not be minimal at all. Take

$$g_0(x) = 4, \qquad g_1(x) = 2x + 2, \qquad g_2(x) = x^2 + x + 1.$$

Then $M=(4,\ 2x+2,\ x^2+x+1)$ contains $2g_2(x)-xg_1(x)=2$, hence $g_0(x)$ is not minimal. We may try to avoid the difficulty by imposing further restrictions upon the coefficients a_{ki}, but the conditions obtained are too complicated to be of any practical value.

3. We shall describe now a different type of normalization for the system $g_k(x)$ which does not have the defect of the trivial normalization (6). Suppose that $g_0(x), \cdots, g_m(x)$ are minimal in M and let $0<k\leqq m$. It follows from (1) that $q_kg_k(x)-xg_{k-1}(x)$ is a polynomial of degree $<k$; hence by (3),

$$q_kg_k(x) - xg_{k-1}(x) = \sum_{i=0}^{k-1} b_{ki}g_i(x). \tag{7}$$

Conversely, if we are given polynomials

$$g_k(x) = a_kx^k + \sum_{i=0}^{k-1} a_{ki}x^i, \quad (a_k \neq 0,\ k = 0, \cdots, m),$$

which satisfy a recursion of the type (7) then they are minimal in $M=(g_0(x), \cdots, g_m(x))$.

To show this, consider the set of polynomials of the form

$$f(x) = h(x)g_m(x) + \sum_{i=0}^{m-1} b_ig_i(x), \qquad b_i \in R. \tag{8}$$

The multiple by a constant and the sum of two polynomials of the form (8) has again the same form, and also $xg_k(x)$ for $k<m$ by (7); hence the polynomials (8) form an ideal $M'\subseteq M$ which contains every $g_k(x)$, so $M'=M$. It follows that every $f(x)\in M$ can be written in the form (8). Notably if $f(x)$ has the degree $k\leqq m$ then it can be written in the form

$$f(x) = \sum_{i=0}^{k} b_ig_i(x), \qquad b_k \neq 0. \tag{9}$$

Hence the leading coefficient of $f(x)$ is b_ka_k which proves that $g_k(x)$ is minimal in M.

We may assume that

$$0 \leqq b_{ki} < q_k, \qquad (k = 1, \cdots, m;\ i = 0, \cdots, k-1) \text{ in (7).} \tag{10}$$

Otherwise put $b_{ki}=q_kd_{ki}+b_{ki}'$, $0\leqq b_{ki}'<q_k$ and $g_k'(x)=g_k(x)-\sum_{i=0}^{k-1}d_{ki}g_i(x)$ which is obviously minimal in M. We have by (7)

$$q_k g_k'(x) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x) - \sum_{i=0}^{k-1} q_k d_{ki} g_i(x)$$

$$= x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki}' g_i(x) \qquad 0 \leqq b_{ki}' < q_k.$$

Hence, starting from $g_1(x)$ we can exchange the $g_k(x)$ step by step until all of them satisfy (10).

Again b_{ki} and $g_k(x)$ are uniquely fixed by this condition. Suppose that $g_k'(x)$, $k=0, \cdots, m$, are minimal polynomials satisfying (7) and (10) with coefficients b_{ki}'. Obviously $g_0'(x)=g_0(x)$, hence we may assume $g_i'(x)=g_i(x)$ for $i<k$. Then

$$q_k g_k'(x) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki}' g_i(x), \qquad 0 \leqq b_{ki}' < q_k,$$

and by (9), since the leading coefficients of $g_k(x)$ and $g_k'(x)$ are equal,

$$q_k g_k'(x) = q_k\left(g_k(x) + \sum_{i=0}^{k-1} b_i g_i(x)\right) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x) + \sum_{i=0}^{k-1} q_k b_i g_i(x),$$

hence $b_{ki}'=b_{ki}+q_k b_i$, $b_{ki}'=b_{ki}$, $g_k'(x)=g_k(x)$.

Finally we show:

$$q_1 q_2 \cdots q_k g_k(x) = f_k(x) g_0(x), \qquad 0 < k \leqq m, \tag{11}$$

where $f_k(x)$ has the degree k and leading coefficient 1. From (7), $q_1 g_1(x)=(x+b_{10})g_0$ which proves (11) for $k=1$. Assuming induction on k, we have from (7)

$$q_1 \cdots q_k g_k(x) = q_1 \cdots q_{k-1}\left(x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x)\right)$$

$$= (x + b_{k,k-1}) f_{k-1}(x) g_0 + \sum_{i=0}^{k-2} b_{ki} q_{i+1} \cdots q_{k-1} f_i(x) g_0$$

$$= f_k(x) g_0.$$

It follows immediately from (11) and Gauss' lemma that

$$g_0 \equiv 0 \pmod{q_1 \cdots q_m}, \qquad g_k(x) \equiv 0 \pmod{q_{k+1} \cdots q_m} \qquad \text{for } k > 0.$$

We can write $g_0=q_0 q_1 \cdots q_m$, $g_k(x)=q_0 q_{k+1} \cdots q_m f_k(x)$ for $k<m$, $g_m(x)=q_0 f_m(x)$, which implies that q_0 is a common divisor of every $g_k(x)$. Consequently $q_0=1$ and

$$g_0 = q_1 \cdots q_m, \qquad g_k(x) = q_{k+1} \cdots q_m f_k(x), \qquad k < m,$$

$$g_m(x) = f_m(x).$$

Summary: We have shown that M possesses a basis $(g_0(x), g_1(x), \cdots, g_m(x))$ with the following properties:

$$g_0 = q_1 \cdots q_m, \tag{12}$$

$$q_k g_k(x) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x), \tag{13}$$

$$q_k > 0,\ 0 \leqq b_{ki} < q_k, \qquad 0 < k \leqq m,\ 0 \leqq i < k. \tag{14}$$

The integers q_k, b_{ki} are uniquely determined by M and the corresponding $g_k(x)$ are minimal in M. Thus the $g_k(x)$ form a canonical basis of M which does not suffer from the defect of the system in §2. All that remains to be shown is that given an arbitrarily prescribed system of "invariants" (14) we can always determine the $g_k(x)$ from (12) and (13) recursively. Suppose that for $i<k$, $g_i(x)$ has already been determined and turns out to be divisible by $q_{i+1} \cdots q_m$.* Then the right-hand side of (13) is divisible by $q_k q_{k+1} \cdots q_m$ and $g_k(x)$ can be determined from the formula and is divisible by $q_{k+1} \cdots q_m$. Thus there is a one-to-one correspondence between the ideals of $R[x]$ and the systems of integers (14).

4. As a typical application let us determine the number of ideal divisors of degree m of a constant c, that is, the number of different ideals of degree m in $R[x]$ which contain the constant c. Let $q_1 \cdots q_m = d$ be a divisor of c. The number of different systems of invariants belonging to a fixed set of q_i's is, according to (14), $q_1 q_2^2 \cdots q_m^m$. Each of these ideals can be multiplied by a principal ideal (q_0) where $q_0 d$ is a divisor of c. Hence the total number of ideal divisors of degree m of c is

$$N_m(c) = \sum_{q_0 \cdots q_m / c} \prod_{i=0}^{m} q_i^i.$$

If in particular $c = p_1 \cdots p_r$ is square-free, then

$$N_m(c) = \prod_{i=1}^{r} (2 + p_i + p_i^2 + \cdots + p_i^m) = \prod_{i=1}^{r} \left(1 + \frac{p_i^{m+1} - 1}{p_i - 1}\right).$$

5. Turning now to the general case it is clear that if R is an arbitrary pr.i.d., the whole argument of sections 2 and 3 remains valid except that we have to find a substitute for the normalizing conditions (14). Two types of normalizations are required. First, we must find a well defined representative of each class of associated elements of R, and secondly we must have a well defined representative of each class of residues modulo a given (normalized) element of R. In some of the more important pr.i.d.'s this is achieved in a very natural way. For example, if R is the domain of polynomials in one variable and with coefficients in a field F, then each class of associated polynomials has a member whose leading coefficient is 1, and we may regard this as the normalized representative. Also each class of residues modulo a normalized $f(x)$ contains exactly one member whose degree is less than the degree of $f(x)$.

* This is certainly true for g_0 by (12).

Of course the invariants q_k, b_{ki} are now not numbers but elements of R. In the particular case when $R=F[y]$, q_k is a non-zero polynomial in y with leading coefficient 1, and b_{ki} is a polynomial of a lower degree than q_k (*e.g.* 0 if q_k has the degree 0).

If R is not a principal ideal ring, the construction seems to break down completely. It would be particularly desirable to obtain a canonical basis when R is the ring of polynomials with integral coefficients in one or several variables, or the ring of polynomials in more than one variable over a field F. In all these cases there is a unique factorization in R and also the Hilbert basis theorem is valid for the ideals of $R[x]$.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

NOTE ON AN ARITHMETIC FUNCTION

L. CARLITZ, Duke University

1. Put

$$F(a, m) = \sum_{r^s=m} \mu(r)a^s,$$

where $\mu(r)$ is the Möbius function (for references see [**3**, pp. 84–86], [**1**]). When $a=p^n$, where p is a prime, $F(p^n, m)/m$ is the number of (primary) irreducible polynomials in $GF[p^n, x]$ of degree m; it is also familiar [**4**, p. 20] that the number of irreducible polynomials of degree m and belonging to the exponent e is $\phi(e)/m$, where $\phi(e)$ denotes the Euler function and m is a *proper* divisor of $p^{ne}-1$; that is,

$$m \mid p^{ne}-1, \qquad m \nmid p^{nr}-1 \qquad \text{for } 1 \leqq r < e.$$

The fact that m is a divisor of $\phi(e)$ appears incidentally; we wish to point out a simple direct proof. More generally we prove that if e is a proper divisor of a^m-1 then $m \mid \phi(e)$.

Indeed it is evident that $a^m \equiv 1 \pmod{e}$. But since e is a proper divisor of a^m-1, it follows readily that m is the exponent to which a belongs (mod e), and therefore m is a divisor of $\phi(e)$.

We remark that in the special case $a=p$, $e=p^m-1$, this result was noted by U. Scarpis [**3**, p. 139].

The formula

$$F(a, m) = \sum_{d \mid a^m-1}{}' \phi(d), \tag{1}$$

where the summation is restricted to proper divisors of a^m-1, was proved by Dickson for arbitrary a [2, p. 36]. Thus, the result just proved gives another direct proof of $F(a, m) \equiv 0 \pmod{m}$.

2. It may be of interest to mention a property of $\Phi(M)$, the Euler function for $GF[p^n, x]$, the domain of polynomials in x with coefficients in $GF(p^n)$: $\Phi(M)$ is defined as the number of polynomials in a reduced residue system (mod M), $M \epsilon GF[p^n, x]$. Then we may state the following:

THEOREM. *If E is a proper divisor of A^m-1 then $m \mid \Phi(E)$.*

The proof is exactly like the proof of the analogous property of $\phi(e)$, indeed $A^m \equiv 1 \pmod{E}$ is the exponent to which A belongs (mod E), and therefore m is a divisor of $\Phi(E)$.

Corresponding to (1) we now have

$$F(a, m) = \sum_{D \mid A^m-1}{}' \Phi(D), \tag{2}$$

where again the summation is restricted to proper divisors, and

$$a = \mid A \mid = p^{n \deg A}.$$

It is easily verified that Dickson's proof of (1) applies to (2) also.

To indicate an occurrence of $\Phi(E)/m$, we take $A = x$ and recall some notions due to Ore [5, pp. 248–250]. For given $M \epsilon GF[p^n, x]$ there exist polynomials $e(x)$ such that

$$e(x) = \sum_{i=0}^{k} a_i x^{p^{ni}} \equiv 0 \pmod{M} \qquad (a_i \epsilon GF(p^n)); \tag{3}$$

if $e(x)$ is a polynomial of the indicated form with minimum k for which (3) holds, then M is said to *belong* to $e(x)$; the polynomial $E = \sum a_i x^i$ *corresponds* to $e(x)$. Also for $x \nmid E$, the exponent of E is the least $m > 0$ such that $E \mid x^m - 1$; Ore calls m the *index* of $e(x)$. Then Ore has proved that the number of irreducible polynomials of degree m and belonging to $e(x)$ is $\Phi(E)/m$, where m is the index of $e(x)$.

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A CRITERION FOR THE POLYNOMIAL SOLUTIONS OF A CERTAIN RICCATI EQUATION

J. G. CAMPBELL, Albany, Kentucky

Let a Riccati equation have coefficients that are polynomials. The degrees of these polynomials restrict the degrees of possible polynomial solutions in a very definite way. We shall prove the following

THEOREM. *Let the Riccati equation*

$$Ay' = B_0 + B_1y + B_2y^2 \tag{1}$$

have polynomial coefficients A, B_0, B_1, B_2 *of degrees* a, b_0, b_1, b_2 *respectively. If* $a \geqq 1$ *then the degree* m *of any polynomial solution of* (1) *is restricted as follows:*

If $a \neq 1+b_1$, *then* m *must be one of the five numbers,* $1+b_0-a$, $a-b_2-1$, b_0-b_1, $\frac{1}{2}(b_0-b_2)$, b_1-b_2;

If $a=1+b_1$, *then* $b_0-b_1 \leqq m \leqq b_1-b_2$.

There is no implication here that a polynomial solution exists. The theorem states that a polynomial not of one of the degrees stipulated cannot satisfy equation (1).

Proof: The degrees of the four terms in equation (1) are respectively $a+m-1$, b_0, b_1+m, b_2+2m. Two, at least, of these degrees must be equal for the equation to be satisfied by the polynomial y; there must be no single highest degree term. By equating all possible pairs of the four degrees listed, we find that $a=1+b_1$ or m is one of the five numbers listed in the theorem.

If $a=1+b_1$, then the four terms of the equation have degrees b_1+m, b_0, b_1+m, b_2+2m, respectively. Now either $b_1+m<b_0$, or $b_1+m=b_0$, or $b_1+m>b_0$.

If $b_1+m<b_0$, then we must have $b_0=b_2+2m$, so that $m=\frac{1}{2}(b_0-b_2)$.

If $b_1+m=b_0$, then $m=b_0-b_1$.

If $b_1+m>b_0$, then the b_2+2m term cannot stand alone with highest degree, so $b_2+2m \leqq b_1+m$, or $m \leqq b_1-b_2$. Since $\frac{1}{2}(b_0-b_2)$ is the mean of (b_0-b_1) and (b_1-b_2), it lies between them. Thus, if $a=1+b_1$, then

$$b_0 - b_1 \leqq m \leqq b_1 - b_2.$$

This completes the proof of the theorem.

This theorem was suggested by a problem in E. D. Rainville's *Intermediate Differential Equations*, and we borrow an example from that text (p. 42):

$$(9x^4 + 18x^3 + 11x^2 + 2x)y' = -9x^4 - 6x^3 + 4x^2 + (18x^3 + 24x^2 + 4x + 2)y + 3y^2.$$

Here $a=4$, $b_0=4$, $b_1=3$, and $b_2=0$. Thus $a=1+b_1$ and the theorem states that $1 \leqq m \leqq 3$. The above equation has the five polynomial solutions

$$x,\ x^2+2x,\ 2x^2+7x/3,\ -x^2-2x/3-\tfrac{2}{3},\ 3x^3+5x^2+3x.$$

These solutions are of degrees 1, 2, 2, 2, 3, so that each value of m admitted by

the inequality is actually present. Thus the inequalities of the theorem cannot be improved for these values of a and the b's.

Rainville [1] proved that the factorization of polynomial differential operators of order two is dependent on the polynomial solutions of a Riccati equation of the type treated in this note.

The author is grateful to the referee for assistance in putting this note into form for publication.

Reference

1. E. D. Rainville, this MONTHLY, vol. 48, 1941, pp. 519–521.

THE SYLVESTER-FRANKE THEOREM

LEONARD TORNHEIM, University of Michigan

G. B. Price* has indicated the existence of interest in a completely simple and direct proof of the Sylvester-Franke theorem. Here is presented a proof based upon elementary transformations.

Let $A=(a_{ij})$ be an n by m matrix. Let $\lambda(p, t)$ $(p=1, \cdots, {}_tC_s)$ range through the ${}_tC_s$ ways of choosing s numbers from the set $1, \cdots, t$. Let $A_{\lambda(p,\ m)\lambda(q,\ n)}$ be the s-rowed minor of A obtained by using the rows determined by $\lambda(p, m)$ and the columns determined by $\lambda(q, n)$. Then the ${}_mC_s$ by ${}_nC_s$ matrix $A^{(s)}=(A_{\lambda(p,\ m)\lambda(q,\ n)})$ is called an s-th compound of A. If $m=n$ we write $\lambda(p)$ for $\lambda(p, m)$.

SYLVESTER-FRANKE THEOREM. *If $A^{(s)}$ is an s-th compound of the n by n matrix A, then*

$$|A^{(s)}| = |A|^{\epsilon}, \quad \textit{where} \quad \epsilon = {}_{n-1}C_{s-1}. \tag{1}$$

There are three types E_1, E_2, E_3 of elementary transformations: E_1 is the multiplication of a row or column by a constant k; E_2 is the interchange of two adjacent rows or columns; and E_3 is the addition of k times a row or column to another row or column. If B is transformed into C by E_i, then

$$|C| = \mu_i |B|, \tag{2}$$

where $\mu_1=k$, $\mu_2=-1$, and $\mu_3=+1$.

The proof of the theorem will be based on the following lemma.

LEMMA. *Let B be an arbitrary square matrix such that*

$$|B^{(s)}| = |B|^{\epsilon}. \tag{3}$$

If B is transformed into C by an elementary transformation E, then $|C^{(s)}| = |C|^{\epsilon}$.

We state first that

* Some identities in the theory of determinants, this MONTHLY, vol. 54, 1947, pp. 75–90.

$$|C^{(s)}| = \mu_i^s |B^{(s)}|. \tag{4}$$

We shall prove this in detail only for $i=2$; the other two cases are easier. Suppose that the u-th and $(u+1)$-th rows of B have been interchanged to give a matrix C. The minors of C which do not involve the u-th or $(u+1)$-th rows equal the corresponding minors of B. If both u and $u+1$ appear in $\lambda(p)$, then $C_{\lambda(p)\lambda(q)} = -B_{\lambda(p)\lambda(q)}$. The number of rows of $C^{(k)}$ having such minors is ${}_{n-2}C_{k-2}$. Next suppose u appears in $\lambda(p)$ but $u+1$ does not. Let $\lambda(p')$ be obtained from $\lambda(p)$ by replacing u by $u+1$. Then $B_{\lambda(p)\lambda(q)} = C_{\lambda(p')\lambda(q)}$ and $B_{\lambda(p')\lambda(q)}$, $= C_{\lambda(p)\lambda(q)}$ since the same elements are used and in the same arrangement, because the u-th and $(u+1)$-th rows are adjacent. Thus from this cause ${}_{n-2}C_{k-1}$ pairs of rows have been interchanged in going from $B^{(k)}$ to $C^{(k)}$. Thus the total number of changes in sign in going from $|B^{(k)}|$ to $|C^{(k)}|$ is

$${}_{n-2}C_{k-2} + {}_{n-2}C_{k-1} = {}_{n-1}C_{k-1} = e.$$

From (4), (2), and (3) it now follows that

$$|C^{(s)}| = \mu_i^e |B^{(s)}| = \mu_i^e |B|^e = |C|^e, \tag{5}$$

and the proof of the lemma is complete.

The proof of the Sylvester-Franke Theorem can now be given as follows. It is known that there exists an n by n matrix

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad 0 \leqq r \leqq n \tag{6}$$

which can be transformed into the given n by n matrix A by a finite sequence of elementary transformations $E^{(1)}, E^{(2)}, \cdots, E^{(v)}$ as follows:

$$E^{(1)}: \quad D \to D_1, \tag{7}$$

$$E^{(2)}: \quad D_1 \to D_2, \tag{8}$$

$$\cdots\cdots\cdots$$

$$E^{(v)}: \quad D_{v-1} \to D_v = A.$$

The matrix D can be calculated by starting with A and reducing it to the form D by a sequence of elementary transformations. Now $|D| = 1$ if $r=n$ and $|D| = 0$ if $r<n$. Also if $r=n$, $D^{(s)} = I_n C_s$ and $|D^{(s)}| = 1$; otherwise $|D^{(s)}| = 0$. Hence the relation $|D^{(s)}| = |D|^s$ holds in both cases. Then by the lemma, $|D_1^{(s)}| = |D_1|^s$. Thus, the hypotheses of the lemma are satisfied also in (8), and $|D_2^{(s)}| = |D_2|^s$. If follows by induction that $|A^{(s)}| = |A|^s$, and the proof is complete.

Similar methods may be used to give the next results.

COROLLARY. *If A is either equivalent, congruent, or similar to B, then $A^{(s)}$ bears the same relation to $B^{(s)}$.*

Thus to determine the elementary divisors of $A^{(s)}$, find B similar to A and in canonical form and find the elementary divisors of $B^{(s)}$.

COROLLARY. *If the rank of an m by n matrix A is r, then $A^{(s)}$ has rank ${}_rC_s$. If $A^{(s)} \neq 0$, then A is equivalent to B if and only if $A^{(s)}$ is equivalent to $B^{(s)}$.*

COROLLARY. *If A is positive definite, then $A^{(s)}$ is also positive definite. If A is symmetric and $A^{(s)}$ has at least two rows, then A is definite if and only if $A^{(s)}$ is.*

A NOTE ON SCALAR FUNCTIONS OF MATRICES

RICHARD BELLMAN, Stanford University

It is well known that the scalar functions of the matrix A which occur as the coefficients of λ in the characteristic equation of A,

$$|A - \lambda I| = 0 = \lambda^n - f_1(A)\lambda^{n-1} + f_2(A)\lambda^{n-2} - \cdots + (-1)^n f_n(A), \tag{1}$$

possess the important commutative property

$$f_k(AB) = f_k(BA). \tag{2}$$

The purpose of this note is to demonstrate a converse result:

THEOREM. *If $\phi(A)$ is a polynomial in the elements a_{ij}, $i, j = 1, 2, \cdots, n$, of A and has the property that $\phi(AB) = \phi(BA)$ for all square matrices A and B of order n, then $\phi(A)$ is a polynomial in $f_1(A), f_2(A), \cdots, f_n(A)$.*

Proof. Let us consider first the case where A has simple characteristic roots. There exists, then, a non-singular matrix T which reduces A to diagonal form,

$$T^{-1}AT = L = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{3}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of A. Thus,

$$\phi(A) = \phi(T^{-1}LT) = \phi(LTT^{-1}) = \phi(L) = \psi(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{4}$$

where ψ is a polynomial in the λ_i. By choosing T suitably, we may change arbitrarily the order of appearance of the λ_i. From this it follows that ψ is a symmetric function of the λ_i, and hence a polynomial in the elementary symmetric functions of the λ_i, which are precisely the $f_k(A)$.

Since any square matrix may be arbitrarily closely approximated by matrices possessing simple characteristic roots, it follows that the representation obtained above is valid for general square matrices.

LAPLACE TRANSFORM OF $(\operatorname{erf}\sqrt{t})^2$

I. OPATOWSKI, University of Chicago

Let $\operatorname{erf} x = (2/\sqrt{\pi})\int_0^x \exp(-x^2)dx$ and

$$I(t, a) = \int_0^t \exp(-x^2)\operatorname{erf}(ax)dx. \tag{1}$$

In particular $I(t, \infty) = (\sqrt{\pi}/2)\operatorname{erf} t$, $I(\infty, a) = (1/\sqrt{\pi})\arctan a$, $I(t, 1) = (\sqrt{\pi}/4)(\operatorname{erf} t)^2$. The last two relations may be checked by differentiating them with respect to a and t respectively. Using the symbol $L\{F(t)\} = \int_0^\infty \exp(st)F(t)dt$ and expanding $\operatorname{erf}(ax)$ in power series we have:

$$\begin{aligned} L\{I(\sqrt{t}, a)\} &= (1/\sqrt{\pi})\sum_{n=0}^{\infty}[(-)^n a^{2n+1}/n!(2n+1)]L\left\{\int_0^t \exp(-x)x^n dx\right\} \\ &= s^{-1}[\pi(s+1)]^{-1/2}\sum_{n=0}^{\infty}(-)^n(2n+1)^{-1}(a/\sqrt{s+1})^{2n+1} \\ &= s^{-1}[\pi(s+1)]^{-1/2}\arctan(a/\sqrt{s+1}). \end{aligned}$$

The latter series converges only for $-1 < s \geqq a^2 - 1$, but it may be seen directly from equation (1) that the Laplace transform of $I(\sqrt{t}, a)$ exists for any s whose real part is >0. The last equation reduces to a known result* for $a = \infty$:

$$L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}.$$

For $a = 1$ the formula gives:

$$L\{(\operatorname{erf}\sqrt{t})^2\} = (4\operatorname{arccot}\sqrt{s+1})/(\pi s\sqrt{s+1}).$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

Editorial Note. The following papers are a selection from those submitted to Classroom Notes in response to the editor's invitation appended to the paper by W. R. Ransom, "Bringing in Differentials Earlier," this MONTHLY, vol. 48, p. 336.

DIFFERENTIALS

M. K. FORT, JR., University of Illinois

1. Introduction. The beginning student usually has more trouble understanding differentials than any other topic in Calculus. One reason for this, as

* R. V. Churchill, Modern Operational Mathematics in Engineering, New York, 1944.

has often been noted, is the use of the "dy/dx notation" instead of the "$f'(x)$ notation" for derivatives. However, even the use of the superior "$f'(x)$ notation" does not help the student to understand what the author of his text is trying to get across when he comes to the notorious proof that $dx=\Delta x$. In order to eliminate the confusion concerning differentials, it is sufficient (and probably necessary) to first give an adequate treatment of the concept of function. With regard to functions, the average Calculus text is open to the following criticisms:

(1) "Function" is never actually defined, but instead a definition is given for the logically different concept "the variable y is a function of the variable x."

(2) The symbol "$f(x)$" is used to denote both a function and the value of that function at x.

(3) No distinction is made between a function and an equation which is used to define the function. (*e.g.* Text books frequently speak of "the function $y=x^2+3$" rather than "the function f for which $f(x)=x^2+3$.")

We now outline a treatment of functions and differentials which the author feels is suitable for presentation at the Calculus level.

2. Functions. F is a *function* if F is a rule which associates with each member x of a certain set A of objects a unique member $F(x)$ of a set B of objects. The set A is the *domain* of the function F and the set B is the *range* of F. If x is in the domain of F, then $F(x)$ is the *value of F at x*. The sets A and B are often sets of numbers, but this is not always the case.

Example 1. Let S be the rule which associates with each real number x its square x^2. Then $S(x)=x^2$ for each real number x. The domain of S is the set of all real numbers and the range of S is the set of all non-negative real numbers.

Example 2. Let d be the rule which associates with each point (x, y) of the plane the distance from (x, y) to $(0, 0)$. The domain of d is the set of all ordered number pairs and the range of d is the set of all non-negative real numbers. A function whose domain is a set of ordered number pairs is often called a *function of two variables* and the value of such a function f at (x, y) is denoted by $f(x, y)$ instead of $f((x, y))$. Thus for d we have $d(x, y)=(x^2+y^2)^{1/2}$.

Example 3. Let C be the function which associates with each triangle T its inscribed circle $C(T)$. The domain of C is the set of all triangles and the range of C is the set of all circles.

It is convenient to use the expression "ordinary function" to describe a function whose domain and range are both sets of numbers. As soon as derivatives are defined (carefully avoiding the dy/dx notation) the following example is instructive.

Example 4. Let D associate with each differentiable function f the function f'. Then D is a function whose domain and range are both sets of ordinary functions.

3. Differentials. Let f be a function which has a derivative at a number x. We define the *differential of f at x* to be the function g for which $g(h)=f'(x)\cdot h$ for

each real number h. Thus the differential of f at x is an ordinary function which is linear and whose domain is the set of all real numbers. We shall denote the differential of f at x by df_x. The *differential of* f is a function of two variables whose domain is the set of all pairs (x, h) for which x is a number at which f has a derivative and h is a real number; the differential of f is denoted by df and is defined by the equation $df(x, h) = df_x(h)$. (If the reader chooses, he may replace h by Δx throughout this paper.)

If one defines in the usual way the addition of functions, the multiplication of functions and multiplication of functions by numbers, then it is easy to prove that

$$\begin{aligned} d(f+g) &= df + dg \\ d(f\cdot g) &= df\cdot g + f\cdot dg \\ d(c\cdot f) &= c\cdot df \end{aligned}$$

whenever f and g are functions and c is a number.

We define the identity function I by letting $I(x) = x$ for each real number x. It is easy to prove that $dI_x(h) = h$ for all numbers x and h. Hence, for any function f which has a derivative at x,

$$df_x(h) = f'(x)\cdot dI_x(h)$$

for all h. It follows that the function df_x is equal to the product of the function dI_x by the number $f'(x)$ and thus

$$df_x = f'(x)\cdot dI_x.$$

The above equation is true for each x at which f has a derivative, and hence

$$\text{(i)} \qquad df = f'\cdot dI.$$

Formula (i) is equivalent to the theorem "$dy = f'(x)dx$" which states that the differential of a function f is equal to the product of the derivative of f and the differential of the identity function.

The differential df is a function of two variables. In order to check his understanding of our notation the reader should prove the following statement. If f'' exists, then the partial derivative of df with respect to the first variable exists and is equal to the differential of f'.

4. Integration. The Editor has requested discussion of the following question:*

(ii) Does the "dx" of differential calculus have the same meaning as the "dx" in the integral $\int_a^b f(x)dx$?

The answer to this question is both yes and no.

If $\int_a^b f(x)dx$ is the Riemann integral (*i.e.* defined as the limit of a finite sum), then there is no connection between the "dx" in the integral and "dx" as used in

* This MONTHLY, vol. 58, 1951, p. 337.

the differential calculus. In this case the "dx" in the integral is used to make the "x" a bound variable. It serves the same purpose that the first "j" does in the expression "$\sum_{j=3}^{5} k^j$." (Notice that if we omitted the j and instead wrote $\sum_3^5 k^j$ then we would be in doubt as to whether $3^j+4^j+5^j$ or $k^3+k^4+k^5$ was intended.) While $\int_a^b f(x)$ would not be a good notation for the integral (since x appears in this expression as a free variable), there is no logical objection to a notation such as $\int_a^b f$. However, it would be difficult to express $\int_2^3 (x^2-5)dx$ in this latter notation since there is no commonly used notation which denotes the function whose value at x is x^2-5. This function can be denoted by an expression such as $\lambda x(x^2-5)$ [see The Calculi of Lambda-Conversion†], and if this notation were used for the function then we could denote our integral by $\int_a^b \lambda x(x^2-5)$ instead of $\int_a^b (x^2-5)dx$. This concludes the discussion of our answer of "no" to (ii).

We now examine the circumstances under which the answer to (ii) is "yes." Let us suppose that f is a continuous function. Then it is possible to prove that there exist functions g such that $g'=f$. It follows from (i) that $dg=f\cdot dI$ for such functions g. We may now define $\int f\cdot dI$ to be the class of all functions g for which $dg=f\cdot dI$. (The "dI" in $f\cdot dI$ is actually superfluous since it is just as reasonable to denote this class of functions by $\int f$.) Finally, we can define $\int_a^b f\cdot dI=g(b)-g(a)$ where g is a member of the class $\int d\cdot dI$. Thus, if integration is introduced via the indefinite integral or primitive method, and "dx" represents the differential of the identity function, then it is possible to interpret the "dx" in $\int_a^b f(x)dx$ as being the same as the "dx" of the differential calculus.

5. Conclusion. The author feels that most of the confusion concerning differentials disappears if the instructor:

(1) uses the word "variable" only to mean a symbol (*i.e.* an ink spot or chalk mark, etc.,) and avoids the usual implication that a variable is a mysterious Dr. Jekyll and Mr. Hyde type of gadget;

(2) carefully distinguishes between a function f and the number $f(x)$ which is the value of f at the number x;

(3) defines the differentials of functions and not the differentials of variables.

THE RELATION OF DIFFERENTIAL AND DELTA INCREMENTS

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Do mathematicians in general have as much trouble understanding the use and meaning of the differential notation as is indicated by the sporadic references to it which have appeared in this MONTHLY? It is to be hoped not, for there should be no confusion *when the symbolism is properly defined.*

This note is written in response to the Editor's two queries: (1) what is a differential and (2) how is the differential notation related to that for integration? One consistent set of definitions is given here which answers the questions in the order they are proposed.

† Alonzo Church, *The Calculi of Lambda-Conversion*, Princeton University Press, 1941.

Let $y=F(x)$ be a continuous function with continuous derivatives of all orders. The derivative of this function is found by the usual limiting process and the result may be indicated by any one of the usual symbols, say $D_x y$.

Let (x, y) and (x', y') be two different solutions of the above equation. We define the delta-increments as

$$\text{(1a)} \qquad \Delta x = x' - x \quad \text{and} \quad \Delta y = y' - y.$$

Then we define the differentials of these variables as

$$\text{(1b)} \qquad dx = \Delta x \quad \text{and} \quad dy = (D_x y)\Delta x.$$

The first definition in (1b) applies specifically to the independent variable and the second to the dependent one. Furthermore, these definitions are consistent for, if we write $y=x$, the second definition yields the same result which is stated in the first.

From the latter two definitions we obtain the usual forms

$$\text{(2a)} \qquad dy = (D_x y)dx \qquad \text{and} \qquad \text{(2b)} \qquad \frac{dy}{dx} = D_x y.$$

The second of these forms supplies the reason for writing the derivative in its usual fractional form. It also shows the change from Δx to dx gives a symmetry to the notation.

The first of these forms has two interpretations:

(i) it is a *rule* for carrying out the operation of "taking the differential"; and

(ii) it is a *formula* for computing the value of the differential of the dependent variable.

The right member of (2a) is a function of two independent quantities, x and dx. If dx is held constant, dy becomes a function of x alone to which we can apply the rule in (i) and define the second differential of y as

$$\text{(3a)} \qquad d(dy) = d^2 y = [(D_x^2 y)dx]dx = (D_x^2 y)dx^2.$$

Further, we define the higher differentials of y as the result of successive applications of this rule under the same conditions and obtain

$$\text{(3b)} \qquad d^n y = (D_x^n y)dx^n.$$

These differential equations may then be transformed into the usual fractional notation for derivatives of higher orders.

If the function $F(x)$ is expanded into a Taylor's series, we may write

$$\text{(4a)} \qquad \Delta y = (D_x y)dx + \frac{1}{2!}(D_x^2 y)dx^2 + \cdots + \frac{1}{n!}(D_x^n y)dx^n + \cdots.$$

By the aid of our definitions this series may be transformed into

$$\Delta y = dy + \frac{1}{2!}d^2y + \frac{1}{3!}d^3y + \cdots + \frac{1}{n!}d^ny + \cdots . \tag{4b}$$

Thus it is seen that our definitions are convenient as well as consistent.

It is well-known that within the interval of convergence the partial sums of the terms of this series may be used as successive approximations to Δy. For instance, the first term alone gives the value Δy would have if its rate of change were constant while x changed from x to x'. The first two terms yield the value Δy would have if this rate of change itself had changed at a constant rate. The first three terms have the constant rate removed one step further. And partial sums with more terms have corresponding interpretations.

If the function $F(x)$ does not possess continuous derivatives beyond the n-th order, we can of course obtain no more than the n-th differential. At the same time, the series approximation would likewise not extend beyond the n-th order.

The extension of these definitions to functions of more than two variables is illustrated by two examples. Let $z=G(x, y)$. As independent variables, we define the differentials of x and y as equal to their delta-increments. Then

$$dz = G_x dx + G_y dy, \tag{5}$$

where the subscript denotes partial differentiation. This form gives both a *rule* for "taking the differential" and a *formula* for computing its value. The definition may be extended to any order. Furthermore, the relation between Δz and its differentials of all orders is given by a form identical with that in (4b).

If $z=G(x, y)$ where $y=F(x)$, the value and form of dz is still given by (5) wherein the value of dy is given by (2a). The second differential, however, is more complicated; it is

$$d^2z = G_{xx}dx^2 + 2G_{xy}dxdy + G_{yy}dy^2 + G_y d^2y. \tag{6}$$

The differentials of higher orders are correspondingly more complicated but can with care be written down. Also, the form in (4b) still holds true.

Turning now to the notation for integration, we find its connection with differential notation dependent upon the meaning attached to the integral.

(a) We may begin by defining the integral sign as the indefinite operator defined in the equation

$$\int dy = y + C.$$

If $y=F(x)$ and $D_x y=f(x)$, we have by substitution that

$$\int f(x)dx = F(x) + C.$$

By this definition, integration is defined (except for the constant C) as the inverse operation of "taking the differential." Hence, the integral sign as an

operator in this sense has no meaning unless it is applied to a perfect differential.

(b) Alternately, we may define Δy exactly by the expression

$$\Delta y = \int_{x_0}^{x_0+\Delta x} f(x)dx,$$

where $y=F(x)$ and $D_x y=f(x)$ as above.

The value of this integral may be approximated by taking $f(x)=f(x_0)$ throughout the interval. This approximate value is $(dy=)f(x_0)\Delta x$. A better approximation may be obtained by dividing the interval into more and smaller parts and using the above approximation for each part. The limit as the number of parts becomes infinite is the exact value of the integral.

This method of approximating Δy does not correspond, except for the first step, with the method which uses either form of (4). Hence the dx in this integral has no relation either to Δx or to the previous form of dx.

The only obvious reason for using the same symbolism in (b) as in (a) is that the definition in (a) may be used to evaluate the limit represented by (b), namely as

$$F(x_0 + \Delta x) - F(x_0).$$

Thus by writing the definite integral in the same form as the indefinite, the known *properties of the latter form* may be used to find the *value of the former*. As with differentials, one interpretation gives the form, the other gives the substance.

TOWARD UNDERSTANDING DIFFERENTIALS

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1. Introduction. The mathematics teacher hardly needs to excite himself today about identification of differentials with "nascent" or "evanescent" increments or with "vanishingly small" quantities, or even with "infinitesimals." If physics has not entirely abandoned these mystical ideas, at least mathematics has done so. It is now standard mathematical pedagogy to introduce differentials as follows. Let $y=f(x)$, where x is the independent variable. Then we define $dx\equiv\Delta x$ and $dy\equiv f'(x)dx$, where Δx is an arbitrary, non-zero increment in x. A picture goes with these definitions, and we explain to our students that a perfectly good value for dx is the distance in millimeters to the remotest known star; dy/dx is still equal to $f'(x)$.

(Of course, if we are as careful as we should be, we say that dx is a *variable* increment rather than an *arbitrary* increment, for we need to "see" the "dx" in most cases—certainly in integrands, where to give dx an arbitrary value would

be to introduce ambiguity: $\int 6x^2 = 2x^3 + c$ because I took $dx = 1$; $\int 6x^2 = x^3 + c$ because Professor Smith took $dx = 2$.)

But it is time that we stopped patting ourselves on the backs for having invented a good definition. We need now to attack the problems of *understanding*, which our logical definition has by no means solved—even, I think, for most of us teachers. These problems are the problems of concreteness; they are, frankly, psychological; and they arise when we interchange dependent and independent variables, introduce a new variable, or become involved with relations between several variables—especially when more than one are independent.

To be sure, our elementary text-books prove that dy/dx is invariant under each of the first two of these changes. (The proofs, however, could be improved: If x is the independent variable, we should logically write $(dx)_x$ and $(dy)_x$ for the symbols defined above, and we should use corresponding notations when other variables are chosen as independent. We should then show that the ratio, $(dy)_t/(dx)_t$ does not depend on the choice of the independent variable t—including of course the special cases $t = x$ and $t = y$. Thereafter we could feel logically secure in writing dy/dx without subscripts.) But there remains the haunting if purely psychological question of just which "d" is the "Δ" in a given differential expression. (And are we not indeed expecting our freshmen to comprehend something a shade worse than a variable *functional*?) Since concreteness seems to be a *sine qua non* for mental tranquility, we might perhaps agree to identify the differential with the increment for, say, the *last* variable whose differential appears in any specific problem—and keep it that, no matter what other variables we may later introduce. This is quite justifiable for problems involving a single independent variable, and something like it may be optimum pedagogy. *Note, however, that it actually dispels the mystery of the unspecified differential increment only at the expense of introducing the inelegance of the specified differential increment.* Also, it necessitates further artificialities in approximation problems in which one would normally identify some *other* increment with a value of the corresponding differential.

The problem of concreteness is more difficult in the case of several independent variables. For example, if $z = f(x, y)$, where x and y are independent variables, we customarily define (or *should* define) $(dx)_{x,y} \equiv \Delta x$, $(dy)_{x,y} \equiv \Delta y$, and $(dz)_{x,y} \equiv f_x(x, y)(dx)_{x,y} + f_y(x, y)(dy)_{x,y}$, where Δx and Δy are independently variable increments in x and y, respectively. If we now choose to regard two other parameters, u and v (whether they are among x, y, and z or not), as the independent variables, we arrive, by means of certain simple and elegant theorems to be found in any respectable calculus text-book, at a formal identity which may be expressed as $dz = f_x(x, y)dx + f_y(x, y)dy$. (How many of us teachers have stopped to think that this means, in particular, that $\Delta z = f_x(x, y)(dx)_{y,z} + f_y(x, y)\Delta y$?) Again we may answer the yearning for concreteness by a device: we may agree that the last *two* differentials which appear in any specific problem shall be interpreted throughout that problem as independent increments—*if there are precisely two independent variables in the problem.* But if x and y are functions of

two, or one, or a million independent variables, just so long as $z=f(x, y)$, those same admirable text-book theorems show us that $dz=f_x(x, y)dx+f_y(x, y)dy$. And when not only *what* the independent variables are, but *how many of them* there are, is unspecified, I am afraid that our wish for concreteness will have to remain not entirely satisfied. (As a test of our *own* need for concreteness, we might read critically the paragraphs in current elementary differential equations text-books devoted to the "total" equation $P(x, y, z)dx+Q(x, y, z)dy+R(x, y, z)dz=0$ and to its geometric applications.)

It seems to me that, in seeking a way out of our dilemma, we should frame our definitions with reference to neither *increments* of variables nor any *particular set* of independent variables. I offer the following general, yet skeletal and condensed development with the hope that it may achieve for differentials (A) continued dissociation from notions of smallness, (B) symmetry of definition, (C) stability of definition under transformations, (D) a degree of compatability with the psychological desire for concreteness of interpretation, and (E) creatively critical consideration at levels of one and two independent variables by teachers who wish to make the concept more meaningful.

2. An algebraic development of differentials. (The justification for this title will appear in due course.) In an attempt to minimize prejudice in this theoretical discussion, we avoid the "d" notation; this we accomplish by reserving Latin letters for the primary variables and using the corresponding Greek letters for their differentials. We assume that all of our functions are "sufficiently well-behaved."

Let all of the relations* between a set of variables† $x_i (i=1, 2, \cdots, l)$ be implied by a set of r independent equations

$$f_m(x_1, x_2, \cdots, x_l) = 0 \qquad m = 1, 2, \cdots, r. \tag{1}$$

Since the rank r of the matrix

$$(f_{mx_i}) \tag{2}$$

equals both‡ the number of independent equations in (1) and the number of the x_i which are defined by (1) as functions of the others, we have $r<l$. Hence the r equations in the new variables ξ_i,

$$f_{mx_1}\xi_1 + f_{mx_2}\xi_2 + \cdots + f_{mx_l}\xi_l = 0, \tag{3}$$

have solutions other than the trivial one. Indeed, since the subsets x_j' $(j=1, 2, \cdots, r)$ of the x_i for which determinant $(f_{mx_j'}) \neq 0$ coincide with the subsets‡ of the x_i which are defined by (1) as functions of the others, their posi-

* By "relations" we shall mean "functional relations."

† When each variable literal subscript is introduced, we state its range; thereafter this range for that subscript is assumed.

‡ P. Franklin, A Treatise on Advanced Calculus, New York, 1940, pp. 340–347.

tions coincide with those of the subsets ξ_j' of the ξ_i which are defined by (3) as functions of the remaining ones of the ξ_i.

The differentials of the x_i relative to the system (1) we define to be the (corresponding) ξ_i as linearly interrelated by (3).

Let

$$f_0(x_1, x_2, \cdots, x_l) = 0 \tag{4}$$

represent *any* relation between the x_i. Since (4) is implied by (1), the rank of the matrix

$$(f_{nx_i}) \qquad n = 0, 1, 2, \cdots, r$$

is the same as that of (2), namely r. Hence all sets of values of the ξ_i which satisfy (3) also satisfy

$$f_{0x_1}\xi_1 + f_{0x_2}\xi_2 + \cdots + f_{0x_l}\xi_l = 0. \tag{5}$$

In particular, if all of the relations between the x_i are implied by the set of $\bar{r}$ independent equations

$$f_p(x_1, x_2, \cdots, x_l) = 0 \qquad p = r+1, r+2, \cdots, r+\bar{r}, \tag{6}$$

then

$$f_{px_1}\xi_1 + f_{px_2}\xi_2 + \cdots + f_{px_l}\xi_l = 0. \tag{7}$$

Now if we require of the ξ_i only that they satisfy (7), the arguments by which we deduced (7) from (1) and (3) permit us to deduce (3) from (6) and (7). Thus, the differentials of a set of variables are dependent only on the relations between those variables and not on any particular system which expresses those relations; and so *we may omit the phrase* "relative to the system (1)" *in our earlier definition.*

Certain other invariant properties of systems which define the relations between the x_i may be deduced from the reciprocal implications of (1) and (6). Thus: *we have* $\bar{r}=r$, since the rank of (f_{qx_i}) $(q=1, 2, \cdots, r+\bar{r})$ is equal at once to r and $\bar{r}$; *if* determinant $(f_{mx_j'}) \neq 0$, *then* determinant $(f_{px_j'}) \neq 0$, *and conversely*, since these conditions express the solvability of (1) and (6), respectively, for the x_j'; hence also *if* (3) *is solvable for a set* ξ_j' *of the* ξ_i, then (7) is solvable for the ξ_j', and conversely. We repeat here that, *if* (1) *is solvable for a set* x_j' *of the* x_i, *then* (3) *is solvable for the corresponding* ξ_j' *and conversely.*

Thus for any set of related variables x_i $(i=1, 2, \cdots, l)$ there are a number $r<l$ and a set of subsets $x_j'(j=1, 2, \cdots, r)$ which are defined as functions of the remaining ones of the x_i, say x_k'' $(k=1, 2, \cdots, l-r)$; the differentials ξ_j' of the x_j' are then necessarily defined as linear functions of the differentials ξ_k'' of the x_k''. These subsets and these definitions are provided by *any* system of the form (1) which expresses all of the relations between the x_i and by the associated system of the form (3).

3. Transformation and constraint of variables. Transformation of variables amounts to (i) the addition to (1) of certain equations which involve new variables as well as the old ones and (ii) the addition to (3) of corresponding equations which involve new differentials and primary variables as well as the old ones, with (iii) the rank of the new matrix corresponding to (2) maintained equal to r. Neither the previous definitions of the original differentials, nor the previously established relations between them, nor any previous concept or interpretation (see Section 5) of them need be changed in any way. (To be sure, we must envision spaces of more dimensions *when* we wish to regard the augmented system of primary variables and differentials as a whole.)

Constraint of variables, whether with or without addition of new ones, involves the addition to (1) and to (3) of new equations with an increase in the rank of the matrix corresponding to (2). The differentials are thus also constrained. But (1) and (3) continue to hold, and we remain, although restricted, within the configurations in terms of which we may have interpreted those relations (see Section 5).

4. Two explicit formulas. For simplicity in the following we shall use the x_j' and ξ_j' notations of Section 2 to indicate the typical sets of r "dependent" variables and of their differentials (that is, those for which (1) and (3) can be theoretically solved), and the x_k'' and ξ_k'' notations to indicate the corresponding $l-r$ "independent" variables and their differentials (those as functions of which (1) and (3) determine the x_j' and ξ_j' respectively). For further brevity we put $s=l-r$. From (3) we have

$$(8) \qquad \textit{If } x_i = \phi_i(x_1, x_2, \cdots, x_l), \textit{ then } \xi_i = \phi_{ix_1}\xi_1 + \phi_{ix_2}\xi_2 + \cdots + \phi_{ix_l}\xi_l.$$

In particular,

$$(9) \qquad \textit{If } x_i = \psi_i(x_1'', x_2'', \cdots, x_s''), \textit{ then } \xi_i = \psi_{ix_1''}\xi_1'' + \psi_{ix_2''}\xi_2'' + \cdots + \psi_{ix_s''}\xi_s''.$$

5. A vectorial interpretation. The equations (1) represent r surfaces in l-space, and the equations (3) identify the differentials ξ_i with the components of a vector in that space which is tangent to each of these surfaces and is therefore tangent to the s-dimensional surface which is their common intersection. (This vector one would of course call *the differential vector of the* x_i.) Equation (4) represents a surface which passes through this intersection, and (5) is merely a statement of the fact that the differential vector of the x_i is tangent to this surface.

6. Differentials and "vanishing smallness." In the setting up of differential equations the mathematical scientist often derives an approximate equation of the form

$$(10) \qquad A_1\Delta x_1 + A_2\Delta x_2 + \cdots + A_l\Delta x_l \approx 0 \text{ for small increments,}$$

where $A_i = A_i(x_1, x_2, \cdots, x_l)$, and then states that

$$A_1 dx_1 + A_2 dx_2 + \cdots + A_l dx_l = 0 \tag{11}$$

—the procedure whose legitimacy is doubtless responsible for the sometime feeling that differentials are quantities, and in fact quantities which are pretty awfully small. Let us show that the procedure is indeed legitimate. First we note that (10) means that

$$A_1 \Delta x_1 + A_2 \Delta x_2 + \cdots + A_l \Delta x_l = o(\sqrt{\Delta x_1^2 + \Delta x_2^2 + \cdots + \Delta x_l^2}). \tag{12}$$

Now, holding all of an arbitrary set of independent variables $x_1'', x_2'', \cdots, x_s''$ fixed except one, say x_k'', dividing by $\Delta x_k''$, and letting $\Delta x_k''$ tend to zero, we have

$$A_1 \psi_{1 x_k''} + A_2 \psi_{2 x_k''} + \cdots + A_l \psi_{l x_k''} = 0, \tag{13}$$

where $x_i = \psi_i(x_1'', x_2'', \cdots, x_s'')$. Multiplying (13) by ξ_k'', summing the results with $k = 1, 2, \cdots, s$, and using (9), we get (11).

Conversely, (11) implies (10)—a fact of great value in the theory of approximation. This is an immediate consequence of the substitution into (11) of the approximate values $\xi_i \approx \Delta x_i$ which are obtained from (9) when we put $\xi_k'' = \Delta x_k''$ (which, note, we are privileged but not compelled by definition to do) and take account of the formula for approximate increments.

7. Differentials and integration. The integration problem relevant to our discussion is this: Given ξ_i, to find x_i. The solution is denoted by $x_i = \int \xi_i$. Let us illustrate.

First we examine the most familiar case, $y = \int f(x)\xi$. This means that $\eta = f(x)\xi$, and the solution is $y = F(x)$ where $F'(x) = f(x)$, by (9). The method of change of variable amounts to introducing a new relation $x = \phi(t)$, which gives rise to $\xi = \phi'(t)\tau$. In terms of t, our problem becomes that of finding y for which $\eta = f[\phi(t)]\phi'(t)\tau$, and the solution is $y = \Phi(t)$ where $\Phi'(t) = f[\phi(t)]\phi'(t)$.

Consider next the line integral $l = \int (y\xi + x\eta)$ along a path $x = \phi(t)$, $y = \psi(t)$. We are to find l for which $\lambda = y\xi + x\eta$. The normal procedure in line integration would be to find ξ and η from the equations of the path and express our problem in the form $\lambda = f(t)\tau$, but because of the special nature of the integrand (it is an "exact differential") we may expedite matters by introducing a new variable $z = xy$. For then $\zeta = y\xi + x\eta$ and our problem becomes merely to solve $\lambda = \zeta$, with the answer $l = z + c$.

The preceding illustrations involve only one independent variable, but perhaps they are adequate in as tentative a sketch as this one.

EDITORIAL

After reading the numerous papers submitted to Classroom Notes on differentials, and after discussions with other mathematicians, your editor is convinced that there is no commonly accepted definition of differential which fits all uses to which this notation is applied.

In the theory of functions of a single variable, $y = f(x)$, a reasonable case

can be made for the customary definition: $dx=\Delta x$; $dy=f'(x)dx$. This breaks down, however, when one extends it to functions of several variables and considers double integrals of the form $\iint f(x, y)\,dxdy$. Students are rightly baffled when they attempt to convert such an integral to polar coordinates and are told that no longer is it permissible to put $dx=-r\sin\theta\,d\theta+\cos\theta dr$, *etc.* The Jacobian must be used instead, and at this point the logical structure which was built so carefully collapses entirely. If we wish to make calculus an intellectually honest subject and not a collection of convenient tricks, it is time we made a fresh start.

Consideration of the uses of differentials in various parts of calculus indicates that mathematicians really do use these symbols to mean quite different things in different circumstances. It seems to the writer that we should admit this from the outset and not try to give a single definition. But surely we must give our students some explanation of differentials or they will be more confused than ever. The following remarks are an attempt to do this. The best description of a differential that I have been able to formulate is:

DIFFERENTIALS: *The presence of a differential in a mathematical expression is a sign to the reader that this expression is obtained by a limiting process from a second expression in which the differentials dx, dy, etc. are replaced by finite increments* Δx, Δy, *etc. The nature of this limiting process depends upon the particular expression involved and must be inferred from the context.*

This description applies nicely to derivatives where

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

There seems to be no good reason to give dy and dx independent meanings, and no need to write expressions such as $dy=f'(x)dx$. An important use of this formula occurs in error problems, but these are more clearly presented by noting that for small values of Δx,

$$\frac{\Delta y}{\Delta x}\approx f'(x). \qquad [\approx\text{means approximately equal}]$$

Hence

$$\Delta y\approx f'(x)\Delta x.$$

For integrals the description again applies to tell us that $\int f(x)dx$ is by definition the $\lim\sum_{i=1}^{n} f(x_i)\Delta x_i$. Similarly the notation $\int f(x)\,dg(x)$ for a Stieltjes integral tells us what limit has been taken, but does not ascribe an independent meaning to $dg(x)$. The meaning of differentials as used here has no logical connection with their meaning in derivatives. Such a connection is often obtained

incorrectly in teaching by failure to state clearly the theorem regarding change of variable in an integral:

$$\int_{x_0}^{x_1} f(x)dx = \int_{t_0}^{t_1} f[x(t)]\frac{dx}{dt}dt; \quad x_0 = x(t_0);\ x_1 = x(t_1).$$

The proof of this theorem *does not* consist of the observation that: $dx = (dx/dt)dt$; even though this collection of nonsense may be an aid to the memory, a crutch for poor students, and a convenience for the experienced mathematician.

In multiple integrals differentials play a similar role and serve to indicate the type of limiting process which is under consideration. Further applications of this sort include line integrals such as $\int_c P(x, y)dx + Q(x, y)dy$ in which the differentials have a similar meaning but one which is still logically separate from those mentioned above.

Another appearance of these strange creatures is in differential equations such as:

$$xdx + ydy + zdz = 0 \tag{1}$$

whose "solution" is said to be: $x^2+y^2+z^2=c$. Here their interpretation is even more obscure. Any (good or bad) physicist will tell you that (1) means that for small Δx, Δy, Δz;

$$x\Delta x + y\Delta y + z\Delta z \approx 0, \tag{2}$$

where Δx, Δy, Δz are arbitrary changes of these variables consistent with the problem at hand. But clearly (1) is not the straight limit of (2). The problem becomes clearer if we connect the point $P(x, y, z)$ to the point $Q(x+\Delta x, y+\Delta y, z+\Delta z)$ by a curve of parameter t, whose value at P is t_0 and at Q is $t_0+\Delta t$. Then by division of (2) by Δt we obtain:

$$x\frac{\Delta x}{\Delta t} + y\frac{\Delta y}{\Delta t} + z\frac{\Delta z}{\Delta t} \approx 0. \tag{3}$$

Now we take limits as $\Delta t \to 0$ and obtain:

$$x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0. \tag{4}$$

The problem required that (2) hold for all permissible displacements (Δx, Δy, Δz) and so for all curves of this type. The omission of dt in the passage from (4) to (1) is then the statement that (4) is true for all such curves. To show that $x^2+y^2+z^2=c$ is a solution of (1) we assume that x, y, and z are arbitrary differentiable functions of a parameter t which satisfy this relation. We then find that:

$$\frac{d}{dt}(x^2 + y^2 + z^2) = 0$$

is the equation (4); and hence conclude that we have a solution of (1).

In differential geometry expressions such as $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ are considered frequently as if ds, du, and dv were well defined quantities. However, a careful inspection of the use of these forms reveals that in every case such a form is really an integrand with the integral sign omitted for convenience. The meanings of the differentials are to be interpreted as if an integral were written explicitly. A similar understanding exists in probability theory when one speaks of the "probability $f(x)dx$" where $f(x)$ is a probability density.

There is a discredited view that differentials are some sort of "infinitely small quantities." Of course, this is nonsense, but there is a germ of truth in it. For differentials do mark the places in mathematical expressions which are held by small, but finite, quantities while some type of limiting process is carried out in which these small quantities tend to zero. To the writer they have no other consistent meaning.

C. B. A.

CORRECTION

Several correspondents have written the editors regarding the paper by J. J. Hart, "A Correction for the Trapezoidal Rule," in this MONTHLY, vol. 59, p. 33. J. P. Ballantine has pointed out that this formula is actually the Euler-Maclaurin expansion (see L. R. Ford, *Differential Equations*, p. 133).

E. G. H. Comfort and Ballantine have noted a computational error in the value of $\int_4^{5.4} \log x\,dx$. The correct value is 1.82784741 instead of 1.82784744. They have also noted that Weddle's rule is more accurate then the proposed rule, in contradiction to the author's statement. Comparative results are:

	Result		*Error*
The Integral	1·82784	74086	
Trapezoidal Rule	1·82765	51387	$-19\ 22699\times10^{-10}$
Simpson's Rule	1·82784	72580	$-\ 1506\times10^{-10}$
Weddle's Rule	1·82784	74073	$-\ 13\times10^{-10}$
Trapezoidal Rule with correction	1·82784	74464	$+\ 378\times10^{-10}$

Hart reports that similar results appear in the M.A. thesis of Ann G. Lutz at the University of Alabama.

CORRECTION

F. ALBERTI, University of Illinois

The following correction should be made in my note "A Theorem on Evolutes" (this MONTHLY, March, 1952, p. 177): In the theorem, replace the phrase "second order contact" by "two point contact," and, at the end of the theorem, add the phrase "provided the circle's right cylinder has three point contact at P."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, N. Y. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1021. *Proposed by W. R. Ransom, Tufts College*

In the "ride and tie" plan (where the first man rides a horse for a stated time and leaves it for a second man, walking, to ride for a like time while the first man walks on) show that the rate of progress is the harmonic mean between the rates of riding and walking, and find for what fraction of the time the horse can rest.

E 1022. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all triangles for which the three sides and one of the altitudes are, in some order, measured by four integers in arithmetic progression.

E 1023. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Massachusetts Institute of Technology*

(1) Given any set of points in the plane, not all coincident with the origin, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 1.

(2) Given any set of points on the unit circle, not the vertices of a regular polygon, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 2.

E 1024. *Proposed by David Gale, Brown University*

Assuming that the four color theorem is true for planar maps containing a finite number of regions, show that it is also true for planar maps containing an infinite number of regions.

E 1025. *Proposed by K. L. Chung, Cornell University*

Show that if

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$$

and

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} b_j$$

for $k=1, \cdots, n$, then

$$\sum_{j=1}^{n} a_j^2 \leqq \sum_{j=1}^{n} b_j^2,$$

equality holding if and only if $a_j=b_j$, $j=1, \cdots, n$.

SOLUTIONS

E 984 [1952, 252]. *Correction*

In solution I the argument of arc cos should be $x/2$ instead of x.

A Slow Ship Intercepting a Fast Ship

E 991 [1951, 699]. *Proposed by C. S. Ogilvy, Syracuse University*

Two ships leave different points at the same time and steam on straight courses at constant but unequal speeds. Find the condition under which the slower ship can intercept (catch) the faster.

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute. If the slower ship intercepts the faster ship then the ratio of the distances traversed is V_S/V_F, where V_S is the velocity of the slow ship and V_F is the velocity of the fast ship. But the locus of a point such that the ratio of its distances from two fixed points is constant, is a circle. Thus if the path of the faster ship intersects this circle it can be intercepted by the slower one.

Editorial Note. The circle is a circle of Apollonius of the two starting points. If θ is the angle between the faster ship's course and the line joining the two starting points, then it is easily shown that the slower ship can set a course to intercept the faster ship if and only if

$$\theta \leqq \arcsin (V_S/V_F).$$

If we have inequality here, then there are two courses that the slower ship may set. The problem becomes more interesting if the sphericity of the earth is taken into account.

Also solved by J. W. Baldwin, Leon Bankoff, J. H. Braun, R. P. Eisinger, Herbert Emich, Ray Jurgensen, J. D. E. Konhauser, D. C. B. Marsh, Prasert Na Nagara, S. T. Parker, L. A. Ringenberg, Azriel Rosenfeld, and the proposer.

Area-and-Perimeter Bisectors of a Quadrilateral

E 992 [1951, 699]. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Draw a straight line which will bisect both the area and the perimeter of a given quadrilateral.

Solution by Michael Goldberg, Washington, D. C. There is an area bisector for every orientation. Therefore the ratio of the two parts of the perimeter varies from below unity to above unity for a half turn of the area bisector. It follows that there are an odd number of area-and-perimeter bisectors unless there are an infinity of them.

For symmetric quadrilaterals, the axis of symmetry bisects perimeter and area. For an isosceles trapezoid, not only the axis of symmetry, but any line through the midpoint of this axis will bisect the perimeter and the area. For parallelograms, any line through the center will do. The following methods apply to all other cases.

Let the lengths of the sides of the quadrilateral be a, b, c, d. Extend the side b for the distance f and the opposite non-parallel side for the distance e to intersect at P. Let a be the side nearest P. Then if the desired bisector cuts d at the distance x from P, and the side b at the distance y from P, the required halving of the perimeter requires that

$$(x - e) + (y - f) + a = \tfrac{1}{2}(a + b + c + d),$$

or

$$x + y = e + f + \tfrac{1}{2}(b + c + d - a) \equiv k_1. \tag{1}$$

Consider the three triangles which have P as a common vertex and whose opposite sides are a, c, and the desired bisector. The area of the latter triangle is required to be a mean between the areas of the other two triangles. The areas are proportional to the products of the two sides adjoining P. Therefore

$$xy = \tfrac{1}{2}(e + d)(f + b) + \tfrac{1}{2}fe \equiv k_2. \tag{2}$$

Solving equations (1) and (2) for x and y we obtain

$$x,\ y = \tfrac{1}{2}k_1 \pm \tfrac{1}{2}\sqrt{k_1^2 - 4k_2}\,. \tag{3}$$

If the desired bisector cuts two adjacent sides of the quadrilateral, then we have

$$x + y = \tfrac{1}{2}(a + b + c + d) \equiv k_3 \tag{4}$$

and

$$xy = A/\sin\theta \equiv k_4, \tag{5}$$

where A is the area of the quadrilateral and θ is the angle at P. In this case we find

$$x,\ y = \tfrac{1}{2}k_3 \pm \tfrac{1}{2}\sqrt{k_3^2 - 4k_4}\,. \tag{6}$$

Since k_1, k_2, k_3, k_4 are obtainable from the given quadrilateral by euclidean

methods, and only square roots are involved in equations (3) and (6), the desired bisector is obtainable by euclidean constructions.

Also solved by I. A. Dodes, M. S. Klamkin, Sam Kravitz, David Mandelbaum, and B. Martin.

Orthogonal Trajectories of a Family of Cycloids

E 993 [1951, 699]. *Proposed by F. M. Carpenter, Colorado School of Mines*

Find the orthogonal trajectories of the family of cycloids

$$x = a(\theta - \sin\theta),$$

$$y = a(1 - \cos\theta).$$

Solution by E. W. Marchand, Eastman Kodak Company, Rochester, N. Y. Differentiation gives

$$-dx/dy = (\cos\theta - 1)/\sin\theta,$$

and elimination of a gives

$$x/y = (\theta - \sin\theta)/(1 - \cos\theta), \tag{1}$$

which defines θ implicitly as a function of x and y. Hence an orthogonal curve obeys the equation

$$dy/dx = (\cos\theta - 1)/\sin\theta. \tag{2}$$

Differentiation of (1) and elimination of dx between (1) and (2) leads to the separable equation

$$dy/y = [\cot(\theta/2) - 2/\theta]d\theta, \qquad \theta \neq 0,$$

which has the solution

$$y = c(1 - \cos\theta)/\theta^2.$$

Substitution in (1) then yields the solution

$$x = c(\theta - \sin\theta)/\theta^2,$$

$$y = c(1 - \cos\theta)/\theta^2.$$

Also solved by E. S. Keeping, O. E. Stanaitis, and the proposer.

An Inequality Involving Factorials

E 994 [1951, 699]. *Proposed by C. S. Venkataraman, Trivandrum, South India*

Prove that $2!4! \cdots (2n)! > (n+1)!^n$.

I. *Solution by A. E. Livingston, University of Oregon.* The inequality is satisfied for $n=2$. Assume that it is true for $2, \cdots, n-1$. Then, by induction,

$$(2n)!\prod_{k=1}^{n-1}(2k)! > (2n)!(n!)^{n-1} = 2n(2n-1)\cdots(n+1)(n!)^n$$

$$> (n+1)^n(n!)^n = (n+1)!^n.$$

We have equality, instead of inequality, when $n=1$.

II. *Solution by Arthur Sturdley, Massachusetts Institute of Technology.* An easy computation reveals that the sequence $\{a_n\}$, where

$$a_n = \frac{2!4!\cdots(2n)!}{(n+1)!^n},$$

is strictly increasing for $n>0$. Since $a_1=1$, the inequality holds for $n>1$.

Also solved by P. M. Anselone, F. Bagemihl, J. W. Baldwin, Leon Bankoff, J. L. Botsford, J. H. Braun, Aaron Buchman, Kenneth Bush, S. A. Foote, C. V. Fronabarger, Robert Gambill, C. B. Germain, Bernard Greenspan, Emil Grosswald, S. W. Hahn, B. A. Hausmann, Vern Hoggatt, Ray Jurgensen, H. Kaufman, P. G. Kirmser, M. S. Klamkin, Milton Lees, D. B. Marsh, Kovina Milosevich, Rodolfo Morales, Prasert Na Nagara, Robert Oeder, L. A. Ringenberg, Azriel Rosenfeld, J. Shipman, M. R. Spiegel, O. E. Stanaitis, C. A. Swanson, Alfred Sylwester, J. A. Tierney, Peter Treuenfels, F. H. Young, and the proposer. Late solutions by F. F. Dorsey, Herbert Emich, Abraham Franck, Charles Pinzka.

Sum of a Finite Series

E 995 [1951, 700]. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Find the sum of the finite series

$$\sum_{n=1}^{N}(n/2^{2n})\binom{2n}{n}.$$

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute. If we define

$$v_n = \frac{n(n-1)}{3\cdot 2^{2n-1}}\binom{2n}{n},$$

then

$$v_{n+1} - v_n = (n/2^{2n})\binom{2n}{n}.$$

Hence the sum of the series is

$$\frac{N(N+1)}{3\cdot 2^{2N+1}}\binom{2N+2}{N+1}.$$

Also solved by J. W. Baldwin, J. L. Botsford, Kenneth Bush, L. Carlitz, Emil Grosswald, Vern Hoggatt, Philip Kirmser, R. Kissling, M. S. Klamkin, L. A. Ringenberg, O. E. Stanaitis, Chih-yi Wang, and the proposer.

Grosswald pointed out that the sum is asymptotic to

$$\frac{2N+1}{3}\sqrt{\frac{N}{\pi}}.$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4470. [1952, 46]. *Correction*

In line 5 read AB, BC, CA instead of BC, CA, AB.

4493. *Proposed by Paul Erdös, University College, London, England*

For each integer n let $\sigma(n)$ be the sum of its divisors. Prove that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}$$

is irrational.

4494. *Proposed by D. J. Newman, Harvard University*

Let $a_1 < a_2 < \cdots$ be a sequence of positive integers, and let ξ be the unending decimal fraction formed by juxtaposing the a's (e.g. if the a's are the primes, ξ would be $0.23571113\cdots$). Prove that if $\sum 1/a_n = \infty$, then ξ is irrational.

4495. *Proposed by M. S. Wertheim, Ithaca, New York*

Find all regular functions of a complex variable for which the mean value theorem holds for all a, b, in the form

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$

where c is a point on the straight line segment ab.

4496. *Proposed by H. D. Grossman, New York City*

Let j and k be integral constants, one positive and the other non-negative; and let $r_1, r_2, \cdots, r_j$; $s_1, s_2, \cdots, s_k$ range over all possible sets of $j+k$ distinct positive integers. Then prove (or disprove) the formula:

$$\sum^{\infty} (1/r_1^4 r_2^4 \cdots r_j^4 s_1^2 s_2^2 \cdots s_k^2) = \frac{(2j+k)!4^j\pi^{4j+2k}}{(2j+1)!k!(4j+2k+1)!}.$$

4497. *Proposed by Jacques Dutka, Rutgers University*

Set down the first n integers, $1, 2, \cdots, n$, in a line and in a second line beneath set down a permutation of them. The number of permutations in the second line which are disjoint with that in the first (no number appears in the same position in both rows) is known to be the integer nearest to $n!/e$. (See, e.g., Whitworth, *Choice and Chance.*)

Suppose now that there are p rows of permutations. How many sets are there in which each row is disjoint with all the others?

SOLUTIONS

Convergence of Series

4415 [1950, 691]. *Proposed by R. P. Boas, Jr. and W. K. Hayman, Brown University*

Find all the values of α and β for which the series

$$\sum_{n=1}^{\infty} n^{\alpha} \sin(n^{\beta})$$

converges.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. Certain special cases are well known. If $\alpha < -1$ the series converges absolutely for any real value of β. If $\beta < 0$, for large n, $\sin n^{\beta} = O(n^{\beta})$, and the series converges for $\alpha+\beta < -1$. For $\beta = 1$, the series converges if $\alpha < 0$.

Hereafter we may assume $-1 \leqq \alpha < 0$ and $\beta > 0$. Put $\nu = -\alpha$, $0 < \nu \leqq 1$ and let $x^{-\nu} \sin x^{\beta}$ be represented by $f(x)$. Using Euler's summation formula in its simplest form (Knopp, *Infinite Series*, 2nd English edition, p. 521) we have

$$\sum_{x=1}^{n} f(x) = \frac{f(1)+f(n)}{2} + \int_1^n \frac{\sin x^{\beta} dx}{x^{\nu}} + \beta \int_1^n \frac{\phi(x) \cos x^{\beta} dx}{x^{\nu-\beta+1}}$$

$$- \nu \int_1^n \frac{\phi(x) \sin x^{\beta} dx}{x^{\nu+1}}, \qquad \phi(x) = -\sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k\pi}.$$

If we observe that $|\phi(x)|<1$, it is obvious that the last integral converges absolutely as $n\to+\infty$. Upon substituting $t=x^\beta$, we see easily that the first integral, as $n\to+\infty$, converges if $\nu+\beta>1$. The question of convergence of the second integral requires further consideration.

Term by term integration of the series $\phi(x)$ is admissible (Bromwich, *Infinite Series*, 1926, ex. 22, p. 144) and the integral may be put in the form

$$\int_1^n \frac{\phi(x)\cos x^\beta dx}{x^{\nu-\beta+1}} = -\sum_{k=1}^{\infty}\frac{1}{2k\pi}\int_1^n \frac{\sin(x^\beta+2k\pi x)dx}{x^{\nu-\beta+1}} + \sum_{k=1}^{\infty}\frac{1}{2k\pi}\int_1^n \frac{\sin(x^\beta-2k\pi x)dx}{x^{\nu-\beta+1}}.$$

Let us denote the first integral on the right side by G_k and the second by G_{-k}. Integration by parts gives

$$G_k = \frac{-\cos n^\beta}{(\beta n^{\beta-1}+2k\pi)n^{\nu-\beta+1}} + \frac{\cos 1}{\beta+2k\pi} - \int_1^n \frac{[\beta\nu x^{\nu-1}+(\nu-\beta+1)2k\pi x^{\nu-\beta}]\cos(x^\beta+2k\pi x)dx}{(\beta x^\nu+2k\pi x^{\nu-\beta+1})^2}.$$

If it is observed that the first term is of the form $O(n^{-\nu})$ and the integrand of the form $O(x^{-\nu-1})$, the convergence of G_k for $\beta>0$, as $n\to\infty$, is apparent and $G_k=O(1/k)$ is a corollary.

The convergence of G_{-k} may be shown and the corollary $G_{-k}=O(1/k)$ obtained in the very same way if $\beta<1$. To avoid zero denominators in case of $\beta>1$ we write

$$G_{-k} = \int_1^{x_1} + \int_{x_1}^n = G'_{-k}+G''_{-k}$$

where the integer x_1 satisfies $(2k\pi)^{1/(\beta-1)}<x_1<n$. Then, just as in the case of G_k, we have

$$G''_{-k} = \frac{-\cos n^\beta}{(\beta-2k\pi n^{1-\beta})n^\nu} + \frac{\cos x_1^\beta}{(\beta-2k\pi x_1^{1-\beta})x_1^\nu} - \int_{x_1}^n \frac{[\beta\nu x^{\nu-1}-(\nu-\beta+1)2k\pi x^{\nu-\beta}]\cos(x^\beta-2k\pi x)dx}{(\beta x^\nu-2k\pi x^{\nu-\beta+1})^2}.$$

Thus the convergence of G''_{-k}, as $n\to\infty$, is established. Since G'_{-k} is finite, the convergence of G_{-k} for $\beta>0$ and $n\to\infty$ is evident.

From the obvious form

$$G_{-k} = \int_1^n \frac{(2k\pi-\beta x^{\beta-1})\sin(x^\beta-2k\pi x)dx}{2k\pi x^{\nu-\beta+1}} + \int_1^n \frac{\beta x^{\beta-1}\sin(x^\beta-2k\pi x)dx}{2k\pi x^{\nu-\beta+1}},$$

upon integrating by parts the first integral on the right we obtain

$$G_{-k} = \frac{1}{2k\pi}\left[\frac{\cos n^\beta}{n^{\nu-\beta+1}} - \cos 1 - (\beta-\nu-1)\int_1^n \frac{\cos(x^\beta - 2k\pi x)dx}{x^{\nu-\beta+1}} + \beta\int_1^n \frac{\sin(x^\beta - 2k\pi x)\,dx}{x^{\nu-2\beta+2}}\right].$$

Now as $n\to\infty$, the first integral on the right converges if $\nu-\beta+1>0$, and is $O(1)$ as $k\to\infty$. Writing the second integral as

$$\int_1^n x^{2\beta-\nu-2}\sin x^\beta \cos 2k\pi x dx - \int_1^n x^{2\beta-\nu-2}\cos x^\beta \sin 2k\pi x dx,$$

noticing that $\int_1^n x^{2\beta-\nu-2}\sin x^\beta$(or $\cos x^\beta$)dx converges as $n\to\infty$ if $\nu-\beta+1>0$, and using a well known theorem on Fourier coefficients, we obtain easily that the second integral is also $O(1)$. Hence $G_{-k}=O(1/k)$ if $\nu-\beta+1>0$.

Since $\sum O(1/k)/k$ is convergent, the convergence of the integral, as $n\to\infty$, has been established if $\nu-\beta+1>0$. For $\nu-\beta+1=0$ the integral oscillates finitely.

For $\nu-\beta+1<0$ the integral is divergent. In fact a necessary and sufficient condition

$$\left|\sum_{k=x_1}^{m}\frac{G''_{-k}}{k}\right| < \epsilon$$

for the convergence of the integral cannot be satisfied if k and x_1 are properly chosen. To fix the ideas, let k and x_1 be the same as given above in G''_{-k}. If we observe that $2(k+p)\pi x_1^{1-\beta}<1+p/k$, where p is an integer less than k, we see from the above expression for G''_{-k} as $n\to\infty$ that $G''_{-k-p}=O(x_1^{-\nu})$. Since the harmonic series diverges, it is evident that the sum in the above condition can be made larger than an arbitrarily small number ϵ. The reason for the behavior of the series for $\nu-\beta+1<0$ or for $\beta>2$ may be made plausible if we observe that the zeros $x=(2k\pi)^{1/(\beta-1)}$ of the function $N=x^\beta-2k\pi x$ as $k\to\infty$ have a limiting point.

To summarize, in case $-1\leqq\alpha<0$ and $\beta>0$ the series converges if both

(1) $\beta-\alpha>1$, (2) $\alpha+\beta<1$.

If $\beta-\alpha\leqq 1$ and $\beta<1$, or if $\alpha+\beta\geqq 1$ and $\beta>1$, the series diverges.

Integers Equal to the Sum of the Cubes of their Digits

4423 [1951, 42]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine systems of numeration such that there exist pairs of consecutive three-digit integers each of which equals the sum of the cubes of its digits. (For example, 370 and 371 in the decimal scale.)

Solution by P. Na Nagara, College of Agriculture, Bangkok, Thailand. Let the digits of the smaller number be x, y, z in the r-scale. The case $z=r-1\neq y$ must be excluded since

$$x^3 + (y+1)^3 + 0 = x^3 + y^3 + (r-1)^3 + 1,$$

or $3y(y+1)=(r-1)^3$, has only the trivial solution $r=1$, as the largest of the three factors in the left-hand member is $\leqq r-1$. Similarly the case $y=z=r-1$ must be excluded.

From $x^3+y^3+(z+1)^3=x^3+y^3+z^3+1$ we get $z=0$. Now by hypothesis the number equals the sum of its digits: $xr^2+yr+0=x^3+y^3+0^3$ or

$$y^3 - yr - x(r-x)(r+x) = 0. \tag{1}$$

Thus y is a factor of $x(r-x)(r+x)$. If $x=ay$, (1) reduces to a Pell equation and for each rational value of a there are infinitely many solutions. In particular, for $a=1$, this equation and its solutions are

$$(2r+1)^2 - 8x^2 = 1,$$
$$x = y = [(3+\sqrt{8})^n - (3-\sqrt{8})^n]/2\sqrt{8},$$
$$r = [(3+\sqrt{8})^n + (3-\sqrt{8})^n - 2]/4,$$

with $n>1$, $z=0$. ($r=8$: 660, 661 provides an example.)

The substitutions $ay=r-x$ and $ay=r+x$ lead also to Pell equations from which additional solutions may be obtained. No general formula seems likely. The simplest cases occur with (a) $y=r-x$ which results in $r=3x+1$, with $y=2x+1$, $z=0$, and (b) $2y=r+x$ whence we obtain $r=(3k+1)^2$, $x=(k+1)(3k+1)$, $y=(2k+1)(3k+1)$, $z=0$, k arbitrary.

Also solved (partially) by E. A. Franz, and the Proposer. (For the Proposer's solution see *Mathesis*, t. LIX (1950) pp. 300–303.)

A Functional Equation

4424 [1951, 42]. *Proposed by Rufas Isaacs, the Rand Corporation, Santa Monica, California*

Describe the most general continuous real-valued function $f(x)$ $(-\infty \leqq x \leqq +\infty)$ which satisfies

$$f[1 - f(x)] = 1 - f(x).$$

Solution by N. J. Fine, University of Pennsylvania. More generally, consider a fixed continuous $h(x)$ on the (extended) reals, and the corresponding functional equation

$$f(h[f(x)]) = h[f(x)]. \tag{1}$$

The most general continuous solution f is given by the following construction.

Let R be any closed interval (possibly degenerate or infinite) which is mapped into itself by h, that is, $h(R)\subseteq R$. Let f be any continuous function with range R such that $f(x)=x$ on $h(R)$. Then f satisfies (1) for if x is arbitrary, $f(x)$ is in R, $h[f(x)]$ is in $h(R)$, and (1) follows. Conversely, if f satisfies (1) and R is the range of f, then R is a closed interval. For any x in $h(R)$ there is a y such that $x=h[f(y)]$, and (1) implies $f(x)=x$; hence $h(R)\subseteq R$ and f acts like the identity on $h(R)$.

In the problem as stated, h is the involutory homeomorphism $x\to 1-x$, with fixed point $\frac{1}{2}$. The condition $h(R)\subseteq R$ implies $R\subseteq h^{-1}(R)=h(R)$, that is, $h(R)=R$. Thus R is any closed interval symmetric about $\frac{1}{2}$, and f is any continuous function with range R which leaves each point of R fixed.

Also solved by C. M. Ablow, Richard Arens, R. C. Buck, Aaron Herschfeld, J. B. Kelly, A. K. Kruse, C. E. Langenhop and H. D. Block, G. Lumer, G. Marsaglia, D. R. Morrison, M. J. Norris, C. S. Ogilvy, William Scott, E. B. Shanks, Samuel Skolnik, R. W. Wagner, S. S. Walters, Robert Weinstock, J. G. Wendel, and the Proposer.

Editorial Note. Morrison notes that, if continuity is not demanded of the function $f(x)$ which satisfies $f[1-f(x)]=1-f(x)$, $f(x)$ may be constructed as follows. Let S be any set of real numbers symmetric about $\frac{1}{2}$, and let

$$f(x) = x, \qquad \text{if } x \in S;$$

$$f(x) = y \qquad \text{where } y \text{ is any element of } S, \text{ if } x \notin S.$$

Sum of Two Squares and Sum of Two Triangular Numbers

4425 [1951, 113]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

There are infinitely many triangular numbers (numbers of the form $k(k+1)/2$) which are also perfect squares, *viz.*

$$[(17+12\sqrt{2})^n+(17-12\sqrt{2})^n-2]/32.$$

Find numbers which are at the same time a sum of two squares and a sum of two triangular numbers.

I. *Solution by G. W. Walker, Buffalo, N. Y. and E. P. Starke, Rutgers University.* Let a, b, x, y be any integers satisfying

$$\frac{1}{2}x(x+1)+\frac{1}{2}y(y+1)=a^2+b^2, \tag{1}$$

which can be written in the form

$$\begin{aligned}(2x-2a+2b+1)(2x+2a-2b+1)-1 \\ =(2a+2b-2y-1)(2a+2b+2y+1)+1=4K.\end{aligned} \tag{2}$$

If the four factors in (2) are identified respectively as u, v, r, s, we have

$$uv = 4K + 1, \qquad rs = 4K - 1, \tag{3}$$

Conversely, if K is an arbitrary integer and integers u, v, r, s are determined which satisfy (3), then

$$\begin{aligned} a &= (r + s - u + v)/8, & x &= (u + v - 2)/4, \\ b &= (r + s + u - v)/8, & y &= (s - r - 2)/4 \end{aligned} \tag{4}$$

will satisfy (1), as may be verified directly.

It is easy to show that integral values of r, s, u, v satisfying (3) always give integral solutions for a, b, x, y. To get all the basic solutions with no duplications, we may require that $a \geqq b \geqq 0$ and $x \geqq y \geqq 0$; which are equivalent to the following restrictions on the spread of the usable factors

$$\text{(I)} \qquad s + r \geqq v - u \geqq 0, \qquad\qquad \text{(II)} \qquad v + u \geqq s - r \geqq 2.$$

For solutions in positive integers, the first equality sign in (I) and the second in (II) must be eliminated.

II. *Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pennsylvania.* The necessary and sufficient condition that N be a sum of two triangular numbers is that $4N+1$ be a square or a sum of two squares. (The degenerate case in which one of the two triangular numbers is zero occurs if and only if N is itself triangular and $4N+1$ is prime.) This follows at once from the equivalence of the two equations

$$N = \tfrac{1}{2}x(x + 1) + \tfrac{1}{2}y\,(y + 1), \qquad 4N + 1 = (x + y + 1)^2 + (x - y)^2.$$

If now N also is a sum of two squares, a^2+b^2, then it is one of the numbers required by the problem.

The well known criterion that a number be a sum of two squares is that it be of the form $2^\alpha MQ^2$, where $\alpha \geqq 0$, $M > 1$; M, Q odd; every prime divisor of M is of the form $4j+1$.

Three special cases are worth notice. (a) If $y = x+1$, then there results a degenerate solution $a = y$, $b = 0$. If however $y = p^2+q^2$, then $N = y^2 = (2pq)^2 + (p^2-q^2)^2$ and a non-degenerate solution exists.

(b) If X_i are the cited numbers which are both triangular and square, then $N = X_i + X_j$ meets the requirements for any choice of i, j.

(c) Consider $N = \frac{1}{2}x(x+1)$. If $x = s^2$, then $N = [\frac{1}{2}(s-1)s]^2 + [\frac{1}{2}s(s+1)]^2$. If $x = \frac{1}{2}r(r+1)$, then $N = \frac{1}{2}r(r+1) + \frac{1}{2}(x-1)x$. Thus: *If x is a square, N is a sum of two squares, and, if x is a triangular number, N is a sum of two triangular numbers.* Therefore, for x equal to any X_i in (b), N is one of the desired numbers.

Also solved by D. H. Browne, L. A. Ringenberg, J. E. Sanders, F. Underwood, and the Proposer.

Editorial Note. For proof of the assertion in the proposal, see 3677 [1935, 573], and E 954 [1951, 568]. Typical of the many special solutions submitted are the following:

a	b	x	y
$n+1$	$2n+3$	n	$3n+4$
$2n+1$	$5n-1$	$3n+1$	$7n-2$
$3m$	m^2-4	m^2-4	m^2+4
a	a^2-n^2	a^2-n^2+n	a^2-n^2-n
$2mn$	n^4+4m^4	$4m^2n^2+n^4-4m^4$	$4m^2n^2-n^4+4m^4$
m^2-n^2	$m^4+6m^2n^2+n^4$	$(m+n)^4-8m^2n^2$	$(m-n)^4-8m^2n^2$

It is obvious that in using any such formulas any negative x may be replaced by $-x-1$, any negative a by $-a$, (similarly for y, b), and a and b may be interchanged, as may x and y.

Lower Bound to the Maximum Modulus

4426 [1951, 113]. *Proposed by D. J. Newman, Harvard University*

Let a, b, c, d, all have modulus unity. Prove that in the unit circle the polynomial

$$P(z) = az^3 + bz^2 + cz + d$$

has a maximum modulus not less than $\sqrt{6}$. What is the result for a polynomial of degree greater than three?

Solution by G. P. Henderson, University of Western Ontario, London, Ontario. We will prove the following generalization of the stated inequality. If

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_m z^m$$

is a polynomial with $a_n \neq 0$, $a_m \neq 0$, $n > m$, then the maximum value of $|P(z)|$ for $|z| \leqq 1$ is not less than

$$\sqrt{\sum_{r=m}^{n} |a_r|^2 + 2|a_m a_n|}.$$

We are going to give z certain values with $|z| = 1$, and for these we may suppose that $m = 0$, since P can be replaced by P/z^m with no change in absolute value. Since $|z| = 1$,

$$|P(z)|^2 = \sum_{r=0}^{n}\sum_{s=0}^{n} a_r \bar{a}_s z^{r-s}.$$

If ω is a primitive nth root of unity,

$$\sum_{t=0}^{n-1} |P(\omega^t z)|^2 = \sum_{r=0}^{n}\sum_{s=0}^{n} a_r \bar{a}_s z^{r-s} \sum_{t=0}^{n-1} \omega^{t(r-s)}.$$

Now the last sum has the value n if $r-s\equiv 0 \pmod{n}$ and vanishes otherwise. We have then

$$(1/n)\sum_{t=0}^{n-1} |P(\omega^t z)|^2 = \sum_{r=0}^{n} |a_r|^2 + a_0\bar{a}_n z^{-n} + \bar{a}_0 a_n z^n.$$

With $z^n = a_0|a_n|/|a_0|a_n$, the right member becomes

$$\sum_{r=0}^{n} |a_r|^2 + 2|a_0 a_n|.$$

We see that for at least one value of t,

$$|P(\omega^t z)| \geqq \sqrt{\sum_{r=0}^{n} |a_r|^2 + 2|a_0 a_n|},$$

from which the above result follows.

Also solved by M. S. Robertson, G. Szegö, J. L. Ullman, and the Proposer.

Related Absolute Values

4427 [1951, 113]. *Proposed by Paul Erdös, University College, London, England*

Let

$$f_n(x) = \prod_{i=1}^{n} (x - x_i), \qquad -1 \leqq x_i \leqq 1.$$

Prove that there cannot exist numbers a, b such that

$$|f_n(a)| \geqq 1, \qquad |f_n(b)| \geqq 1, \qquad -1 < a < 0 < b < 1.$$

Solution by Robert Steinberg, University of California, Los Angeles. The proof depends on the following lemma: *If* $-1<-c<0<b<1$, *and* $f(x)=|x+c|^b|x-b|^c$; *then* $f(x)<1$ *for* $-1\leqq x\leqq 1$.

To prove the lemma note that, if $-1\leqq x\leqq -c$, then $f(x)$ has its maximum value at $x=-1$, and this maximum value, $(1-c)^b(1+b)^c$, is less than 1. ($1+b<e^b$ and $1-c<e^{-c}$ imply $(1-c)^b(1+b)^c<e^{-bc}\cdot e^{bc}=1$.)

Similarly $f(x)<1$ if $b\leqq x\leqq 1$.

In the interval $(-c, b)$, differentiation shows that $f(x)=(x+c)^b(b-x)^c$ assumes its maximum value of $b^b c^c$ for $x=b-c$; and this again is less than 1.

Now let a and b be any numbers such that $-1<a<0<b<1$. Then setting $a=-c$, we have

$$|f_n(-c)|^b |f_n(b)|^c = \prod_{i=1}^{n} |c + x_i|^b |b - x_i|^c < 1$$

according to the lemma. Since b and c are positive, this implies that at least one of $|f_n(-c)| = |f_n(a)|$ and $|f_n(b)|$ is less than 1.

Also solved by N. J. Fine, L. C. Hsu, and G. W. Walker.

Editorial Note. Walker makes use of a geometrical argument. In Euclidean n-space let the point P have coordinates x_i and consider the 2^n disconnected regions, one in each compartment of coordinates, represented by $|f_n(c)| \geqq 1$ for $c=0$. The inequality $-1 \leqq x_i \leqq 1$ represents all points on and within an n-dimensional hypercube. The effect of varying $c(-1 \leqq c \leqq 1)$ is a rigid translation of the 2^n-sheeted hypersurface along the line joining the origin to $(1, 1, \cdots, 1)$. Of course we get the same effect by translating the hypercube with respect to the hypersurface. It is not difficult to see that there are no points common to the regions bounded by the hypercube and the hypersurface for $0 \leqq c \leqq 1$ which are also common to the hypercube and the hypersurface for $-1 \leqq c \leqq 0$. Thus there exists no set of coordinates $x_i(-1 \leqq x_i \leqq 1)$ for which both $|f_n(a)| \geqq 1$ and $|f_n(b)| \geqq 1$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Conformal Mapping. By Zeev Nehari. New York, McGraw-Hill, viii+396 pp. $7.50.

The object of this book is to give a student with a minimum amount of background (advanced calculus) an introduction to the Dirichlet problem and the conformal mapping problem for simply and multiply connected domains. The author has accomplished this goal and the book fills a definite need for a textbook on conformal mapping on this level. The book should make an excellent text for advanced undergraduate and beginning graduate work especially for persons interested in applied mathematics. Numerous exercises of above average type are given and these should go a long way toward helping the student understand the theory and application of the theory of conformal mapping. The whole book is written in a rather informal or classroom style and this should appeal to a student.

The first chapter of the book is devoted to the statement of the Dirichlet problem and the introduction of auxiliary functions needed for the statement and solution of the problem. The second, third and part of the fifth chapters give the usual material of a beginning course in complex variables. The fourth chapter gives results on closure properties of families of functions of a complex variable. The fifth chapter gives the solution of the problem of mapping one

simply connected domain conformally upon another simply connected domain neither of which is the entire finite plane. The corresponding problem for the closure of a domain with a simple closed curve for a boundary is also considered. The remainder of the fifth chapter is devoted to the application of these results to the Dirichlet problem and to special mapping problems. The sixth chapter gives the mapping properties of many special classes of functions. The seventh and final chapter considers the conformal mapping problem for multiply connected domains.

On the whole the proofs and explanations given in the book are presented in a neat manner. A few statements in the book need some clarification. As a typical example, in Cauchy's theorem it is stated that two simple curves with the same end points determine a simple closed curve.

E. J. MICKLE
Ohio State University

Finite Matrices. By W. L. Ferrar. Oxford University Press, 1951. vi+182 pp. $4.00.

This is a companion volume to the author's *Algebra* which was published in 1941. Many of the basic theorems concerning matrices are assumed with a foot-note reference to the earlier volume. After an introductory chapter, equivalence over a field and equivalence of λ-matrices over a field are considered, but the field is by definition a subfield of the complex field. This third chapter contains the elementary divisor theory. The term *c*-equivalence is introduced for similarity, and in Chapter IV (entitled *Collineation*) the complex canonical form of a matrix under similarity transformations is obtained. From this the rational canonical form is obtained, valid only for a subfield of the complex field.

Chapter V is devoted to infinite series and functions of matrices, and regarding this material the author claims in his preface that "only patches of it here and there appear to have been published before." It is not clear to the reviewer if Ferrar is familiar with Schwertfeger's monograph *Les Fonctions de Matrices*, Paris, 1938. This is certainly a connected treatise. At any rate Ferrar's Chapter V does not follow Schwertfeger very closely, and contains some interesting new material.

Chapter VI contains a treatment of the congruence of matrices, orthogonal and unitary matrices, quadratic and hermitian forms. The reviewer bridled a bit at the definition of a nilpotent matrix as one whose square is zero, but of course Wedderburn's definition is not patented. In Chapter VII some simple types of matric equations are considered.

Chapter VIII, entitled *Miscellaneous Notes*, contains an exposition of the resolvent matrix, *i.e.*, the theory of the Frobenius covariants; positive definite quadratic forms whose variables are subject to a linear condition; and the anti-commutative matrices of Eddington and Newman.

Sets of exercises such as are to be found in *Algebra* are absent from the present book, which makes it less convenient for use as a text book.

The printing and paper are excellent, but my copy may have suffered from the Atlantic passage because its board covers are not even developable surfaces.

C. C. MacDuffee
University of Wisconsin

NEW BOOKS RECEIVED

Teaching Mathematics in the Secondary School. By L. B. Kinney and C. R. Purdy. New York, Rinehart and Company, Inc., 1952. xvi+381 pages. $5.00.

Mathematics, its Magic and Mastery. Second edition. By Aaron Bakst. New York, D. Van Nostrand Co., Inc., 1952. 14+790 pages. $6.00.

Chemical Calculations. By S. W. Benson. New York, John Wiley and Sons Co., 1952. xi+217 pages. $2.95.

The Nature of Number, An Approach to Basic Ideas of Modern Mathematics. By Roy Dubisch. New York, The Ronald Press Company, 1952, xii+159 pages. $4.00.

Tables of the Exponential Function e^x, Applied Mathematics Series 14. By the National Bureau of Standards Computation Laboratory. Washington, Government Printing Office, 1951. x+540 pages. $3.25.

Colloque de Geometrie Differentielle. (Louvain, 1951) Georges Thone, Liège and Masson et Cie, Paris, 1951, 240 pages. 2450 Fr. francs.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

EDITORIAL NOTE

All club reports for 1950–51 submitted before January 1, 1952 have now been published. Club reports for 1951–52 will not be compiled.

The publication of club programs was initiated several years ago as a medium for exchanging ideas of program topics and club projects. A considerable fund of suggestions is now available to club sponsors and program chairmen, and the publication of additional club reports would not seem to serve any useful function.

The Editor appeals for material to make this department a source of aid and inspiration to club activities, particularly on the undergraduate level. He would welcome sundry items such as new club projects, bibliographies of club topics, abstracts of student articles, expository articles, mathematical "curiosa," original limericks and poems, plays, radio skits, descriptions of career opportunities, and other items of interest to undergraduate students.

BOOKS CONTAINING MATERIAL FOR UNDERGRADUATE MATHEMATICS CLUB PROGRAMS

C. B. READ, University of Wichita

At a recent national convention there was revealed what one professor called "a woeful ignorance" about source material for undergraduate club programs. Some of those present seemed to be unaware of the existence of certain books which were considered by others as almost classic in the field.

With the idea of obtaining some composite opinion as to the value of certain books, the author volunteered to make a brief survey. With the cooperation of one or two colleagues a tentative list of thirty-one books was prepared. This included a majority of the books known to this group which seemed to have value as source material for undergraduate club programs. This list was sent to a group of nineteen individuals. Individuals were selected as those personally known by the author as having some interest in sponsoring club programs, or whose names had appeared in the club section of this MONTHLY as being connected with club programs.

The individuals were asked to place the number one before the five books considered to be of the greatest value, a number two before the five books considered of next highest value, and a number 3 before the next five books in order of value. Space was provided for adding additional books with their rating.

The response to the check list was greater than might have been anticipated. Seventeen of the nineteen replied. However, before presenting results certain comments should be made. First, several individuals indicated that they were not familiar with all the books on the list and hence that a failure to rate the book does not mean that the book is poor, merely that the individual making the rating was not familiar with the book. Secondly, it was pointed out that there is some excellent material available in articles in papers and various magazines. This fact is clearly recognized, but the purpose of this study was primarily to rate *books* in the field. A third point was raised that it might be very desirable to include more books dealing with fundamental concepts of mathematics. One individual pointed out that the number of such books on the list submitted was small. Another individual raised the point that he did not consider books on puzzles and riddles as a good source for mathematics club programs, rather the program should tend to widen the member's knowledge and interest in mathematics rather than entertain. Still another says, " . . . serious scrutiny would cause one to wonder whether or not one should recommend said books for the consideration of youths, or recreation . . . many of the books give incorrect impressions and most of them cause the reader to work too hard for what he gets." (This individual feels few people are in a position to rate the books and declined to attempt the task.) There is no question that all of these points of view are of distinct merit, but even if opinions differ it is quite likely that a composite opinion is worth more than that of any single individual.

In the light of these introductory remarks the list of books which follows may still be of value in giving sources of material for undergraduate mathematics club programs. The first thought in compiling this list was to include every book mentioned by any individual teacher. However, one of those responding included 38 books not on the original list, 28 of these being in a foreign language. One other list included several books in German, but pointed out that unfortunately many undergraduates find material in a foreign language unavailable. Accordingly with the idea of shortening the list, no book is included unless it was rated in the first or in the second category by at least one or in the third category by more than one individual. In several cases the publisher has more than one office. The address given is that of the office confirming the price and the fact that the book was or was not in print as of November, 1951. It represents a composite opinion giving the number of times a book was listed as of the greatest value, next highest, and third highest (in groups of five).

The numbers in parentheses indicate respectively ratings received in groups of 1, 2, 3 and the weighted totals from an arbitrary weighting of five points for a rating in group 1, 3 points for a rating in group 2, and 1 point for a rating in the third group. It is recognized that there may be disagreement with this scale, but it does offer at least some basis for assigning rank. It should also be repeated that not all individuals rated five books in each of the various classes.

Ball, W. W. Rouse: Mathematical Recreations and Essays. The Macmillan Company, 2459 Prairie Ave., Chicago 16, Illinois. $3.95 (10, 4, 1——63)

Courant, Richard and Robbins, Herbert: What is Mathematics? Oxford University Press, Inc., 114 Fifth Avenue, New York. $4.75 (10, 3, 1——60)

Kasner, Edward and Newman, James: Mathematics and the Imagination. Simon and Schuster, Inc., 1230 Sixth Avenue, New York. $3.50 (6, 6, 0——48)

Steinhaus, H.: Mathematical Snapshots. Oxford University Press, Inc., 114 Fifth Avenue, New York 11, New York. $5.00 (5, 5, 1——41)

Kraitchik, Maurice: Mathematical Recreations. W. W. Norton and Company, Inc., 101 Fifth Avenue, New York 3, New York. Out of Print. (6, 0, 6——36)

Bakst, Aaron: Mathematics, Its Magic and Mastery. D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3, New York. New edition in preparation, available approximately February, 1952. (4, 4, 3——35)

Northrop, Eugene P.: Riddles in Mathematics. D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3, New York. $3.50 (4, 1, 3——26)

Jones, Samuel I.: Mathematical Nuts. S. I. Jones Company, 1122 Belvidere Drive, Nashville 4, Tennessee. $3.50 (3, 3, 0——24)

Abbott, Edwin A.: Flatland. Barnes and Noble, Inc., 105 Fifth Avenue, New York 3, New York. $2.00 (3, 2, 3——24)

Andrews, William S.: Magic Squares and Cubes. Open Court Publishing Company, Box 402, La Salle, Illinois. Out of Print. (2, 3, 4——23)

Smith, David E. and Ginsburg, J.: Numbers and Numerals. National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W. Washington 6, D. C. $0.75 (3, 2, 2——23)

Licks, H. E.: Recreations in Mathematics. D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3, New York. Out of print. (2, 2, 6——22)

Jones, Samuel I.: Mathematical Wrinkles. S. I. Jones Company, 1122 Belvidere Drive, Nashville 4, Tennessee. $3.50 (1, 5, 2——22)

Jones, Samuel I.: Mathematical Clubs and Recreations. S. I. Jones Company, 1122 Belvidere Drive, Nashville 4, Tennessee. $3.00 (2, 2, 5——21)

Rupert, W. W.: Famous Geometric Theorems and Problems. D. C. Heath and Company, 1815 Prairie Ave., Chicago 16, Illinois. Out of Print. (2, 2, 5——21)

Merrill, Helen Abbot: Mathematical Excursions. Bruce Humphries, Inc., 30 Winchester St., Boston, Massachusetts. Out of Print. (2, 3, 1——20)

White, William F.: A Scrapbook of Elementary Mathematics. Open Court Publishing Company, Box 402, La Salle, Illinois. $1.50 (1, 3, 4——18)

Alliston, Norman: Mathematical Snack Bar. Chemical Publishing Company, 212 Fifth Ave., New York 10, New York. Out of Print. (2, 1, 2——15)

Dudeney, Henry E.: The Canterbury Puzzles. Thomas Nelson and Sons, Edinburgh, Scotland. (Not in stock at 385 Madison Ave., New York 17, New York) $2.00 (1, 3, 1——15)

Heath, Royal V.: Mathemagic. Simon and Schuster, Inc., 1230 Sixth Avenue, New York. (Publisher ignores letters as to availability of this book) (0, 3, 3——12)

Lucas, M. Edouard: Recreations Mathematiques. Libraire Gauthier-Villars, Paris. Out of Print. (1, 1, 1——9)

Row, Sundara T.: Geometric Exercises in Paper Folding. Open Court Publishing Co., Box 402, La Salle, Illinois. $1.00 (0, 3, 0——9)

Phillips, Hubert: Caliban's Problem Book. Thomas De La Rue and Co., Ltd., 84/86 Regent Street, London, England. Out of Print. (1, 0, 2——7)

Dudeney, Henry E.: Modern Puzzles and How to Solve them. C. A. Pearson, London, England. Out of Print. (1, 0, 1——6)

Mott-Smith, Geoffrey: Mathematical Puzzles. Doubleday and Company, Inc., Garden City, New York. Out of Print. (0, 2, 0——6)

Yates, Robert C.: Geometrical Tools. Educational Publishers, Inc., 122 N. Seventh St., St. Louis 1, Missouri. $3.50 (1, 0, 1——6)

Collins, A. Frederick: Fun with Figures. Appleton-Century-Crofts, Inc., 35 West 32nd St., New York 1, New York. Out of Print. (0, 1, 2——5)

Finkel, B. F.: Mathematical Solution Book. Kibler and Company, Springfield, Missouri. (Mail sent to publisher returned—unknown) (0, 1, 2——5)

Sainte-Lague: Avec Des Nombres y Des Lignes. Librairie Vuibert, Boulevard Saint-Germain 63, Paris, France. 590 frs. (1, 0, 0——5)

Kramer, Edna E.: The Main Stream of Mathematics. Oxford University Press, Inc., 114 Fifth Avenue, New York 11, New York. $5.00 (1, 0, 0——5)

Dantzig, T.: Number, The Language of Science. The Macmillan Company, 2459 Prairie Avenue, Chicago 16, Illinois. $4.50 (1, 0, 0——5)

Smith, David Eugene: History of Mathematics (2 volumes) Ginn and Company, 2301 Prairie Avenue, Chicago 16, Illinois. $13.00 (1, 0, 0——5)

Merriman, Gaylord M.: To Discover Mathematics. John Wiley and Sons, Inc., 440 Fourth Avenue, New York 16, New York. $3.25 (1, 0, 0——5)

Bell, Eric T.: Men of Mathematics. Simon and Schuster, Inc., 1230 Sixth Ave., New York 20. New York. $6.00 (0, 1, 1——4)

Schubert, H. C. H.: Mathematical Essays and Recreations. Open Court Publishing Co., Box 402, La Salle, Illinois. Out of Print. (0, 0, 3——3)

Degrazia, Joseph: "Math" is Fun. The Gresham Press, Inc., Eleven Broadway, New York 4, New York. $2.75 (0, 1, 0——3)

Dudeney, Henry E.: Amusements in Mathematics. Thomas Nelson and Sons, Edinburgh, Scotland. (Not in stock at 385 Madison Ave., New York 17, New York.) $2.00 (0, 1, 0——3)

Whitworth, William Allen: Choice and Chance. Stechert-Hafner, Inc., 31 East 10th St., New York 3, New York. $4.00 (0, 1, 0——3)

Levens, Alexander S.: Nomography. John Wiley and Sons, Inc., 440 Fourth Ave., New York 16, New York. $3.50 (0, 1, 0——3)

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. F. E. Bothwell of Northwestern University has accepted a position as a consultant with the United States Naval Ordnance Test Station, China Lake, California.

Mrs. Helen E. Brown, formerly a teacher at Fassifern School for Girls, Hendersonville, North Carolina, is teaching now at Martha Berry School for Girls, Mount Berry, Georgia.

Assistant Professor R. E. Carr of Michigan State College has a position as Research Engineer with North American Aviation, Downey, California.

Associate Professor John Cawley of Lafayette College has retired.

Assistant Professor J. O. Chellevold of Lehigh University has been appointed to a professorship at Wartburg College.

Mr. D. F. Coulter, Jr., formerly a mathematician at the United States Mine Countermeasures Station, Panama City, Florida, has been appointed to an instructorship at Grays Harbor College, Aberdeen, Washington.

Mr. J. G. Cox, Alabama Polytechnic Institute, is now Supervisory Analytical Statistician at Eglin Air Force Base, Florida.

Mr. J. W. Crispin of Wayne University has been appointed Research Associate at the Willow Run Research Center, University of Michigan.

Mr. D. O. Davidson has a position as an engineering assistant at Northrop Aviation, Hawthorne, California.

Research Assistant L. E. Davis of Ohio State University has a position as Mathematician at the Naval Research Laboratory, Washington, D. C.

Mr. A. R. DiDonato, previously a student at Duquesne University, has a position at the DuPont Experimental Station, Wilmington, Delaware.

Associate Professor W. C. Doyle of Rockhurst College has been promoted to a professorship.

Mr. Trevor Evans of the University of Wisconsin has been appointed to an assistant professorship at Emory University.

Dr. A. E. Foster of the University of Kentucky has been appointed to an instructorship at Florida State University.

Associate Professor W. W. Gandy of Texas Agricultural and Mechanical College has a position as Senior Aerophysics Engineer with Consolidated-Vultee Aircraft Company, Ft. Worth, Texas.

Mr. S. I. Gass, formerly a mathematician at Aberdeen Bombing Mission, Los Angeles, has a position now as a mathematician in the Planning Research Division, Directorate Program Standards and Cost Control, United States Air Force, Washington, D C.

Associate Professor Abe Gelbart of Syracuse University is on leave of absence and has been appointed Fulbright Lecturer at the Institute of Technology, Trondheim, Norway.

Mr. L. D. Gregory, University of Oklahoma, has a position as Mathematician at Chance Vought Aircraft Company, Dallas, Texas.

Mr. F. J. Hardy, previously a graduate assistant at Duquesne University, has accepted a position as Engineer at Bell Aircraft Corporation, Niagara Falls, New York.

Mr. H. S. Heaps of the University of Toronto has been appointed to an assistant professorship at Nova Scotia Technical College.

Mr. L. L. Helms, formerly a student at Bradley University, is now a research assistant in the Statistical Laboratory, Purdue University.

Mr. David Horwitz, who has been a student at Illinois Institute of Technology, has a position as Assistant Engineer at Armour Research Foundation, Chicago, Illinois.

Mr. H. V. Huneke of the University of Oklahoma has been appointed Lecturer at the University of Wichita.

Mr. Jiro Ishihara, formerly a student at Northwestern University, is now a staff member of the Digital Computer Laboratory, Massachusetts Institute of Technology.

Dr. W. H. Ito of the University of Minnesota has a position as Senior Project Analytical Engineer at Chance Vought Aircraft, Dallas, Texas.

Mrs. Verba W. Iturralde has been teaching at Bowie High School, El Paso, Texas during 1951–1952.

Mrs. Helen S. Keefer, previously a student at Iowa Wesleyan College, has a position as Engineering Aide with the General Electric Company, Schenectady, New York.

Mr. C. E. Kelley who has been associated with the New York Life Insurance Company has been appointed to an instructorship at the University of Missouri.

Associate Professor E. S. Kennedy of American University of Beirut, Lebanon, has been serving as Fulbright Lecturer at Tehran University, Iran.

Mr. M. R. Kenner, formerly a graduate student at the University of Minnesota, has been appointed to an instructorship at Southern Illinois University.

Associate Professor Fred Kiokemeister of Mount Holyoke College is on leave of absence and has been appointed to a visiting professorship at the University of Chicago.

Mr. S. G. Kneale, previously a graduate student at Harvard University, has accepted a position as Mathematician with the Philco Research Division, Philadelphia, Pennsylvania.

Mr. D. H. Kraft is now Second Lieutenant, United States Air Force; he is serving as a mathematician in the Flight Research Laboratory, Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. Sidney Kravitz has accepted a position as Ammunition Design Engineer with the Picatinny Arsenal, Dover, New Jersey.

Mr. E. R. Lancaster of the University of Detroit has a position as Mathematician with International Business Machines Corporation, Poughkeepsie, New York.

Mr. J. N. P. Lawrence, formerly a student at Vanderbilt University, is now a research assistant at Los Alamos Scientific Laboratory, New Mexico.

Assistant Professor E. L. Lehmann of the University of California at Berkeley has been promoted to an associate professorship; he has been on leave of absence during the year 1951–1952 and has been Visiting Associate Professor at Stanford University.

Assistant Instructor F. H. Lloyd of the University of Missouri has been appointed to an instructorship at Westminster College.

Instructor J. H. Manheim of Wilson Junior College, Chicago, Illinois, has accepted a position as Engineer at the Federal Telecommunications Laboratories, Nutley, New Jersey.

Emeritus Professor H. W. March of the University of Wisconsin has been serving as a special consulting mathematician at United States Forest Products Laboratory, Madison, Wisconsin.

Associate Professor P. E. Meadows of Carroll College is now Principal of Sidney High School, Ohio.

Professor J. N. Michie of Texas Technological College has retired with the title of Professor Emeritus.

Dr. T. S. Motzkin of Hebrew University and the National Bureau of Standards has been appointed to a visiting professorship at the University of California, Los Angeles.

Assistant Professor E. N. Nilson, Trinity College, Connecticut, has been promoted to an associate professorship.

Dr. Ilse L. Novak has been appointed a member of the Institute for Advanced Study.

Dr. Rufus Oldenburger, mathematician-engineer of the Woodward Governor Company, Rockford, Illinois, gave a series of lectures in Paris, France, during the month of April on "Mathematical Engineering Analysis"; these lectures were given to l'Institution des Arts et Métiers, Société Francaise des Mecaniciens, Société des Radioelectriciens, and the Société d'Electronique et d'Automatisme.

Assistant Professor Daniel Orloff of Southern Illinois University has accepted a position as Mathematician at Bell Aircraft Corporation, Niagara Falls, New York.

Mr. S. T. Paine of the Rand Corporation has accepted a position as Mathematician with the Department of Meteorology, University of California, Los Angeles.

Mr. A. H. Payne, formerly a graduate student at the University of North

Carolina, is now a mathematician at Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. G. W. Payne, previously a student at St. Mary's University, is now in military service.

Mr. J. M. Pike who has been a graduate student at Kansas State Teachers College is now Liaison Engineer at the United States Naval Air Missile Test Center, Pt. Mugu, California.

Mr. C. F. Pinzka, previously a statistician with E. I. DuPont de Nemours and Company, New Brunswick, New Jersey, has accepted a position with the Educational Testing Service, Princeton, New Jersey.

Mr. Barth Pollak, formerly a student at Illinois Institute of Technology, has a position as Analyst with the Department of Defense, Washington, D. C.

Dr. R. E. Roberson of the Naval Research Laboratory has accepted a position as Research Engineer with North American Aviation, Downey, California.

Assistant Professor Saul Rosen of Drexel Institute of Technology is now Associate Research Engineer with the Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Mr. Max Rosenberg of the National Bureau of Standards has accepted a position as Mathematician with the Picatinny Arsenal, Dover, New Jersey.

Assistant Instructor J. M. Sandy of the University of Missouri has a position as Mathematician at the United States Naval Ordnance Test Station, Pasadena, California.

Associate Professor C. L. Seebeck of the University of Alabama has been promoted to a professorship.

Mr. D. J. Smith, formerly an Actuarial Trainee at Royal-Liverpool Group of Insurance Companies, New York City, has accepted a position as Engineer with Breeze Corporations, Newark, New Jersey.

Mr. O. S. Spears of the University of Oklahoma is now a member of the Artillery School Board, United States Army, Ft. Sill, Oklahoma.

Mr. R. H. Spohn, previously of the Kellex Corporation, New York City, has accepted a position as Mathematician with the Vitro Corporation of America, West Orange, New Jersey.

Mr. J. R. Swaffield, formerly a student at Boston University, has been appointed to an instructorship at Clark School, Hanover, New Hampshire.

Mr. P. C. Sweetland of Michigan State College has been appointed to an assistant professorship of research at Pennsylvania State College.

Mr. A. D. Talkington of DePauw University is now a mathematician at the National Bureau of Standards, Washington, D. C.

Mr. F. J. Wall who has been a research assistant at the Los Alamos Scientific Laboratory, New Mexico, has accepted a position as Statistician with the Dow Chemical Company, Denver, Colorado.

Dr. L. M. Weiner of the University of Chicago has been appointed to an instructorship at De Paul University for the academic year 1952–1953.

Mr. R. W. Young, Henderson State Teachers College, Arkansas, has been appointed to a professorship at Indiana Technical College.

Associate Professor R. K. Zeigler of the University of California has a position as Statistician with the United States Atomic Commission, Oak Ridge, Tennessee.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and ten persons have been elected to membership by the Board of Governors on applications duly certified.

W. T. ACHOR, Student, Alabama Polytechnic Institute, Auburn, Ala.

R. E. ALLAN, M. A. (Alabama) Ensign, U. S. Navy, Arlington, Va.

J. L. BAKER, JR., M.S. (Northwestern) Asso. Professor, University of Cincinnati, Ohio

D. O. BANKS, B.S. (Oregon S. C.) Grad. Assistant, Oregon State College, Corvallis, Ore.

C. Y. BARTHOLOMEW, B.S. (Rutgers) Junior Research Engineer, National Union Radio Corporation, Orange, N. J.

J. B. BARTOO, M.S. (Iowa) Instr., State University of Iowa, Iowa City, Ia.

C. D. BELL, M.S. (Michigan) Vice President and Actuary, Pioneer Mutual Life Insurance Company, Fargo, N. D.

E. W. BETH, Ph.D. (Utrecht) Professor of Logic, University of Amsterdam, Netherlands

R. E. BLASCH, Student, Hofstra College, Hempstead, N. Y.

I. E. BLOCK, Ph.D. (Harvard) Mathematical Consultant, Philco Corporation, Philadelphia, Pa.

J. R. BOYD, B.S. (Trinity) Teacher, San Marcos High School, Tex.

GERALD BRIGHAM, Student, University of Rhode Island, Kingston, R. I.

IDA BRISKY, Student, Northeastern University, Boston, Mass.

J. L. BROWN, JR., B.S. (Ohio) Asst. Professor, Pennsylvania State College, Pa.

R. W. BROWN, SR., B.S. (Pacific) Teaching Fellow, Oregon State College, Corvallis, Ore.

LUCY J. BULLOCK, M.A. (Boston U.) Asso. Professor, St. Paul's Polytechnic Institute, Lawrenceville, Va.

MARY J. BURNS, Student, Duquesne University, Pittsburgh, Pa.

MRS. LEILA R. H. BUTLER, B.S. (Savannah) Teacher, Walker High School, Savannah, Ga.

G. C. BYERS, B.A. (Michigan) Hancock, Mich.

L. B. CARPENTER, A.B. (Johns Hopkins) Physicist, Army Chemical Center, Md.

N. H. CHOKSY, B.S. in E.E. (Milwaukee Schl. Engg.) Grad. Student, University of Wisconsin, Madison, Wis.

GERALDINE A. COON, Ph.D. (Rochester) Mathematician, Taylor Instrument Company, Rochester, N. Y.

R. C. COURTER, Student, Columbia University, New York, N. Y.

ANN M. CURRAN, M.A. (Connecticut) Asst. Instructor, University of Connecticut, Storrs, Conn.

D. L. DART, Student, Wayne University, Detroit, Mich.

J. R. DAVIS, JR., Student, University of Kentucky, Lexington, Ky.

V. J. DERBES, M.D. (Tulane) Asso. Professor of Internal Medicine, Tulane University, New Orleans, La.

ANDREW DIPRETE, Student, University of Rhode Island, Kingston, R. I.

REV. M. M. DREILING, M.S. (Catholic) Asso. Professor, St. Joseph's College, Collegeville, Ind.

PATRICK DU VAL, Ph.D. (Cambridge, England) Professor, University of Georgia, Athens, Ga.

R. A. EDMONDSON, M.A. (Cornell) Chairman, Division of Natural Science & Mathematics, Wiley College, Marshall, Tex.

BERNARD EPSTEIN, Ph.D. (Brown) Asst. Professor, University of Pennsylvania, Philadelphia, Pa.

H. L. ESTERMAN, Student, University of Rochester, N. Y.

W. J. FLEMING, JR., B.E. (Yale) Asst. Professor of Engineering, University of Buffalo, N. Y.

R. J. FORTIN, Student, University of Rhode Island, Kingston, R. I.

MRS. RUTH M. FRISCH, A.B. (Brown) Grad. Student, Polytechnic Institute of Brooklyn, N. Y.

J. C. FUNNELL, LL.B. (Northeastern) Retired. 4 Union Park, Boston, Mass.

J. J. GEHRIG, B.S. (Wisconsin) Mathematician, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

J. M. GIBLIN, M.S. (John Carroll) Mathematician, Army Map Service, Washington, D.C.

BERNICE GOLDBERG, M.A. (Radcliffe) Mathematician, Cambridge Research Center, Mass.

J. K. GOLDHABER, Ph.D. (Wisconsin) Instr., University of Connecticut, Storrs, Conn.

L. W. GREEN, M.A. (Yale) Assistant, Yale University, New Haven, Conn.

R. O. GROSNICK, B.Ed. (S.T.C., Stevens Point, Wis). Instr., University of Wisconsin, Manitowoc, Wis.

J. J. HAILE, B.S. in Ed. (West Virginia S. C.) Instr., Maryland State College, Princess Anne, Md.

E. E. HAMMOND, JR., Sc.M. (Brown) Lieutenant, U. S. Navy, New York, N. Y.

H. W. HANDSFIELD, A.B. (Columbia) Editor, College Department, McGraw-Hill Book Company, New York, N. Y.

W. C. HANSEN, Student, University of Maryland, College Park, Md.

L. R. HARPER, JR., A.B. (Talladega) Grad. Student, University of Chicago, Ill.

C. R. HEIMLICH, Student, Purdue University, Lafayette, Ind.

H. C. HEWES, JR., Student, Northeastern University, Boston, Mass.

E. E. HIGH, M.A. (Atlanta) Instr., West Virginia State College, Institute, W. Va.

C. G. HILL, A.B. (Morehouse) Instr., Voorhees Junior College. Denmark, S. C.

NANCY HINTZE, Student, University of Oklahoma, Norman, Okla.

KATHERINE A. HOLLAND, B.S. (Alabama Poly.) Mathematician, U.S.N.M.C.S., Panama City, Fla.

W. B. HOLMES, B.S. (Oklahoma) Flight Test Analyst, Boeing Airplane Company, Wichita, Kan.

N. Q. HUBBARD, A.M. (Nebraska) Instr., Lincoln Junior College & High School, Kansas City, Mo.

E. H. INSELMANN, M.A. (Columbia) Mathematician, Frankford Arsenal, Philadelphia, Pa.

LEO KATZ, Ph.D. (Michigan) Asso. Professor, Michigan State College, East Lansing, Mich.

D. A. KEARNS, M.A. (Brown) Asst. Professor, Merrimack College, Andover, Mass.

O. W. LAING, M.A. (Southern California) Instr., Pasadena City College, Calif.

P. F. LARDI, Student, Fordham University, New York, N. Y.

MARGARET M. LASALLE, M.S. (Louisiana) Instr., Francis T. Nichols Junior College, Thibodaux, La.

W. E. LAWRENCE, B.S. (Marquette) Grad. Research Assistant, Marquette University, Milwaukee, Wis.

R. B. LEIPNIK, Ph.D. (California) Asst. Professor, University of Washington, Seattle, Wash.

S. R. LENIHAN, B.A. (U.C.L.A.) Teaching Assistant, University of California at Los Angeles, Calif.

BERNARD LEVENSON, M.A. (Columbia) Research Assistant, Columbia University, New York, N. Y.

G. A. LEWIS, M.A. (N.Y.U.) Instr., Storer College, Harpers Ferry, W. Va.

MRS. HELEN W. LINDLEY, M.S. (Mississippi

S. C.) Grad. Assistant, Tulane University, New Orleans, La.

R. F. LOBER, Student, Alfred University, Alfred, N. Y.

R. G. LONG, Student, Reed College, Portland, Ore.

R. D. LOWE, A.B. (Dickinson) Grad. Assistant, Northwestern University, Evanston, Ill.

G. E. MAHONEY, Student, Boston University, Mass.

E. W. MCCLANE, M.S. (DePaul) Asst. Professor, DePaul University, Chicago, Ill.

D. G. MILLER, B.S. (California) Grad. Student, University of Illinois, Urbana, Ill.

R. C. MILLER, JR., M.A. (Redlands) Mathematician, Naval Ordnance Test Station, China Lake, Calif.

T. O. MOORE, M.A. (Arkansas) Instr., University of Missouri, Columbia, Mo.

P. S. NULL, M.S. (West Virginia) Instr., West Virginia University, Morgantown, W. Va.

J. N. PALMIERI, Student, Brown University, Providence, R. I.

E. H. PRIMOFF, Student, University of California at Los Angeles, Calif.

T. J. REGIMBAL, JR., Student, Gonzaga University, Spokane, Wash.

B. V. RITCHIE, M.S. (Purdue) Instr., University of Detroit, Mich.

D. C. ROSE, M.A. (Kentucky) Instr., University of Kentucky, Lexington, Ky.

LAWRENCE ROSENFELD, M.A. (Harvard) Mathematician, Cambridge Research Center, Mass.

DAVID ROTHMAN, Student, University of Wisconsin, Madison, Wis.

D. A. RUX, B.S. (Marquette) Grad. Assistant, Marquette University, Milwaukee, Wis.

V. G. RYAN, M.A. (Villanova) Instr., St. Joseph's College, Philadelphia, Pa.

D. E. SCHAFER, A.B. (Friends) Medical Student, University of Minnesota, Minneapolis, Minn.

MRS. FARIEBEE P. SELF, M.A. (Louisiana) Asst. Professor, Centenary College, Shreveport, La.

MRS. SEDALIA M. SIMS, B.S. (Wiley) Instr., Wiley College, Marshall, Tex.

SISTER M. ELIZABETH, M.A. (DePaul) Mathematics Department, Alverno College, Milwaukee, Wis.

J. W. H. SPENCER, B.S. (George Washington) Chief, Computing Branch, Army Map Service, Washington, D. C.

H. W. SPRING, JR., Student, University of Rhode Island, Kingston, R. I.

W. C. STONE, M.S. (Chicago) Asst. Professor, Union College, Schenectady, N. Y.

W. E. STRIMLING, M.A. (Minnesota) Teaching Assistant, University of Minnesota, Minneapolis, Minn.

W. C. STYSLINGER, JR., Student, Duquesne University, Pittsburgh, Pa.

ENN TATAR, Student, Wartburg College, Waverly, Ia.

J. T. TATE, Ph.D. (Princeton) Higgins Research Associate, Princeton University, N. J.

L. J. TEUTONICO, Student, St. John's University, Brooklyn, N. Y.

C. W. THOMSON, B.S. (S.T.C., St. Cloud, Minn.) Part-time Instructor, University of Colorado, Boulder, Colo.

B. F. TIFFIN, B.Sc. (Loyola, Montreal) Grad. Student, McGill University, Montreal, Quebec

ELMER TOLSTED, Ph.D. (Brown) Asst. Professor, Pomona College, Claremont, Calif.

H. G. TUCKER, M.A. (California) Grad. Student, University of California, Berkeley, Calif.

R. M. VASSAR, Student, College of Idaho, Caldwell, Idaho

SAMUEL VERBLUNSKY, Ph.D. (Cambridge, England) Professor, Queen's University, Belfast, Northern Ireland

J. A. VOLLKOMMER, Student, St. John's University, Brooklyn, N. Y.

A. W. WALLACE, Student, Northeastern University, Boston, Mass.

L. A. WARE, Ph.D. (Iowa) Professor, State University of Iowa, Iowa City, Ia.

ROBERTA A. WILCOX, Student, University of Rhode Island, Kingston, R. I.

R. R. WILLIAMS, JR., M.S. (Michigan) Chairman, Department of Mathematics, Wiley College, Marshall, Tex.

R. F. WILLIS, A.B. (Columbia) Grad. Student, Columbia University, New York, N. Y.

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the National Bureau of Standards, Washington, D. C., on December 8, 1951. Professor E. R. Sleight, Chairman of the Section, presided at the morning and afternoon sessions.

The meeting was attended by one hundred and twelve persons including the following seventy-five members of the Association:

J. C. Abbott, M. H. Ahrendt, R. P. Bailey, W. E. Barnes, T. J. Benac, E. E. Blanche, J. H. Braun, B. H. Buikstra, W. E. Byrne, Audrey L. Carlan, H. J. Cheston, Jr., Randolph Church, Chester R. Clark, G. R. Clements, J. H. Curtiss, Ruth M. Davis, L. S. Dederick, J. A. Duerksen, William H. Durfee, E. L. Eagle, D. F. Eliezer, P. J. Federico, C. H. Frick, B. C. Getchell, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, R. E. Greenwood, T. N. E. Greville, J. R. Hammond, H. G. Hertz, J. E. Ikenberry, S. B. Jackson, F. E. Johnston, A. E. Landry, Mary A. Lee, D. B. Lloyd, C. J. Maloney, Martin Maltenfort, M. H. Martin, J. M. McLynn, Florence M. Mears, Joseph Milkman, George Millman, W. K. Morrill, C. R. Morris, C. H. Murphy, Jr., W. R. Murray, W. H. Norris, Jr., A. J. Pejsa, G. W. Petrie, III, R. M. Pinkerton, J. W. Popow, C. H. Rawlins, Jr., R. W. Rector, L. V. Robinson, P. C. Rogers, W. G. Saunders, J. B. Scarborough, Paul Shapiro, W. F. Shenton, Sister Gabrielle Marie, E. R. Sleight, J. K. Sterrett, H. C. Stotz, Fedor Theilheimer, O. M. Thomas, P. D. Thomas, J. A. Tierney, Michael Tikson, John Todd, R. E. Walters, M. E. White, and J. W. Wrench.

During the business meeting the Section voted to change the by-laws of the Section making the official name of the Section as follows: The Maryland-District of Columbia-Virginia Section of the Mathematical Association of America. The Section also voted to have a committee appointed to study the suggestions for section activities made at the Minneapolis meeting and report at the spring meeting of the Section. Miss Veryl G. Schult was appointed chairman of this committee. It was announced that Michael Goldberg will act as a member of the executive committee to fill the vacancy made by Professor Klee's leaving the region to go to the Institute for Advanced Study. The members of the nominating committee appointed to submit a slate of new officers at the spring meeting were M. H. Martin, University of Maryland, chairman, W. E. Byrne, Virginia Military Institute, and W. K. Morrill, Johns Hopkins University. It was announced that the invitation to hold the spring meeting at the Virginia Military Institute, Lexington, Virginia, on April 26, 1952 had been accepted by the executive committee. The Section voted unanimously that a letter of thanks be sent to the National Bureau of Standards for their hospitality to the members and guests of the Association during the meeting.

The following papers were presented at the meeting:

1. *The squaring of developable surfaces*, by Mr. Michael Goldberg, Bureau of Ordnance, Navy Department.

The problem of dividing a rectangle into different squares has received considerable attention and many elegant results have been obtained. Mr. Goldberg exhibited some of these results and showed how they may be extended to non-planar surfaces. In particular, the method was applied to

cylinders, Möbius strips and various cones including those which develop into overlapping plane surfaces. Models of these divisions were exhibited.

2. *N-dimensional Hermite polynomials*, by Professor Joseph Milkman, United States Naval Academy.

The definition of N-dimensional Hermite polynomials given by Harold Grad in vol. II, 1949, of *Communications on Pure and Applied Mathematics* is revised to a form suitable for generalization to infinite dimensional spaces and some of their properties including orthogonality developed. A Hermite polynomial of order p is defined to be a polynomial of degree p obtained by the formula:

$$H^p_{i_1 i_2 \cdots i_p} = \frac{(-1)^p}{(2\pi)^p W} \frac{\partial^p W}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_p}}$$

where

$$W = e^{-\pi(x_1^2 + x_2^2 + \cdots + x_N^2)}.$$

Obviously $H^p_{i_1} \cdots {}_{i_p}$ can be expressed as a product of N one dimensional Hermite polynomials.

3. *Partial differential equations and differential operators*, by Dr. L. V. Robinson, Aberdeen Proving Ground.

By making use of the generalized differential operator

$$\lambda = \frac{\partial}{\partial x} + P_1(x, y, z) \frac{\partial}{\partial y} + P_2(x, y, z) \frac{\partial}{\partial z} + \cdots,$$

it was shown how linear partial differential equations of the first and second orders can be solved. Some general structural relations between these differential equations and their solutions were emphasized.

4. *A class of spirals*, by Professor S. B. Jackson, University of Maryland.

In some recent work in gas dynamics, it became important to discuss the geometric behavior as t becomes infinite of curves of the form

$$z = \int_0^t r(u) e^{iu} du$$

where $r(t)$ is a non-negative monotone increasing function, becoming infinite with t. It was shown in this paper that such a curve is an outwinding spiral and that it exhibits one of two characteristics. Either $\lim_{t \to \infty} |z| = \infty$ so that it spirals to infinity in all directions, or the curve lies in a half plane. There is a unique line bounding such a half plane which consists precisely of the limit points of the spiral. Analytic criteria were established to distinguish the two cases.

5. *John Warner's mathematical exercises*, by Professor W. F. Shenton, The American University.

Professor Shenton read and discussed portions of a 32-page pamphlet published in London in 1710. In this pamphlet John Warner, Teacher of Mathematicks, presents the first of a series developing "A New, Plain, and Methodical Institution of Universal Mathematicks Pure and Mixt. Beginning at the very Foundation, and proceeding gradually in the Natural Order those Disciplines ought to be learnt, by such as are minded to go through a Course of this most Useful, Solid Learning. Digested into EXERCISES for the Benefit and Advantage of Learners; which may also contribute somewhat to the Ease and Help of Teachers." Professor Shenton expressed the desire to learn if other numbers of the series had been published.

6. *Use of an electronic analogue machine as a curve fitting device*, by Mr. C. H. Murphy, Jr., Ballistic Research Laboratories, Aberdeen Proving Ground.

An important problem of applied mathematics in the field of aerodynamics is that of determining the coefficients of a differential equation from discrete points of its solution curve. In the field of modern electronic machines precision analogue computers, such as the Reeves Electronic Analogue Computer (REAC), seem to be quite capable of solving the problem. A recent investigation of this use for the REAC was described and graphical results shown. The equation used for the problem was a linear second order complex equation with varying coefficients.

7. *Round table discussion on modern numerical analysis and college mathematics*, by Dr. J. H. Curtiss, Chief, National Bureau of Standards Applied Mathematics Laboratories (Moderator), and Mr. John Todd, Chief, Mr. H. E. Salzer, and Mrs. Ida Rhodes, National Bureau of Standards Computation Laboratory.

Dr. Curtiss began the discussion by stating that with a prospect of fifty new large scale automatic electronic computers to be put into operation within the next five years a need will arise for some two thousand college-trained operators and that from one hundred to two hundred mathematical analysts at the post-Ph.D. level will be needed to insure efficient use of these machines. He further indicated the desirability of mathematics teachers shifting their emphasis from continuous analysis toward those essentially discrete forms of analysis which are used in modern numerical analysis.

Mrs. Rhodes urged that in the undergraduate curriculum more attention be given to errors involved in computing with approximate numbers; calculation using various scales of notation other than the scale with the base ten; theory of polynomial approximation, with special reference to devices for speeding up convergence; modern matrix theory, with special reference to different inversion techniques; asymptotic series; and interpolation, with emphasis on remainder terms.

Mr. Salzer suggested that some of these topics be studied in a one-year course in modern numerical analysis at the junior level (see this MONTHLY, May, 1952) followed by further work at the senior and graduate levels. He further suggested that within the courses now taught in most colleges asymptotic series could be taught while studying integration by parts, difference equations and numerical integration could be taught while studying differential equations, and that field trips and advertising of job openings would stimulate student interest in numerical mathematics.

Mr. Todd discussed the solution of difference equations, showing how some important and interesting results can easily be established which cast light on differential equation theory.

The professional computers urged that only students with the prerequisite qualities of patience, carefulness, and pertinacity be encouraged to take jobs in computing laboratories; but the moderator pointed out that there are opportunities in the research and analytical aspects of computing which do not demand specialized temperamental characteristics. It was suggested from the floor that college teachers could be made to understand the problem best by establishing summer courses or teacher institutes at some of the great computing centers.

C. H. FRICK, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 24, 1951. Chairman for this meeting was Professor P. A. Caris, University of Pennsylvania.

There were fifty-eight present, including the following forty-four members:

Emil Artin, Joshua Barlaz, P. A. Caris, Mary L. Constable, E. H. Cutler, D. J. Davis, J. E. Davis, L. J. Deck, F. L. Dennis, Arnold Dresden, N. J. Fine, H. S. Grant, R. C. Haseltine, Katharine E. Hazard, P. J. Kiernan, J. R. Kline, R. A. C. Lane, V. V. Latshaw, F. L. Manning, Clifford Marburger, S. S. McNeary, A. E. Miller, C. A. Nelson, C. O. Oakley, J. C. Oxtoby, G. E. Raynor, Judith A. Richman, J. W. Ridley, R. D. Schafer, Pincus Schub, M. J. Sendrow, C. A. Shook, J. C. Smith, P. I. Speicher, E. P. Starke, R. R. Stoll, A. W. Tucker, R. N. Van Arnam, N. D. Vlachos, G. C. Webber, Anna P. Wheeler, R. C. Yates, H. M. Zerbe, H. J. Zimmerberg.

At the business meeting the following officers were elected for the coming year: Chairman, F. L. Manning, Ursinus College; Secretary, R. D. Schafer, University of Pennsylvania. The Program Committee for the next meeting will be: C. A. Nelson (Chairman), Rutgers University, G. C. Webber, University of Delaware and Albert Wilansky, Lehigh University. The 1952 meeting of the section will be held at the University of Delaware, Newark, Delaware, on November 29.

The program consisted of the following papers:

1. *The stimulation of interest*, by Colonel R. C. Yates, U. S. Military Academy.

Colonel Yates presented items intended to arouse and maintain student interest in mathematics of the first two college years. Such motivations as linkages, cams, paper folding, elementary solutions of maximum-minimum problems, the properties of the crossed parallelogram, caustics, and trisection were used to give point to the general titles: *Integration with experience*, *Utility*, *The paradox and the puzzle*, *Elegance and simplicity*, *Discovery*, *Speculation*, *Creation*, and *The element of surprise*. Devices and models of various sorts were used to illustrate these means of stimulation.

2. *Constructions with ruler and divider*, by Professor Emil Artin, Princeton University.

A few elementary constructions are given that show that the use of a ruler and divider with fixed opening is equivalent to the use of a ruler and a free divider. The answer to the question: "What geometric problems can be solved by the use of these tools?," is the following:

(a) The problem must be possible with ruler and compass.

(b) The algebraic equation describing the geometric problem must have only real roots for all real values of the parameters that enter in the problem.

The proof of this statement makes use of the theory of real fields. The essential theorems concerning real fields are given.

3. *An infinite-product expansion for analytic functions*, by Professor Bernard Epstein, University of Pennsylvania, introduced by the Secretary.

J. F. Ritt obtained in 1930 the theorem: If $f(z)$ is analytic at $z=0$ and $f(0)=1$, then there exists a uniquely determined set of constants $\{a_k\}$ such that

$$f(z) = \prod_1^\infty (1 - a_k z^k)$$

in some circle $|z| < R$. Evidently R cannot exceed the radius of convergence of the Taylor series of $f(z)$ (about $z=0$) but simple examples show that it may be less. Ritt showed that $R > 1/6C^{-1}$, $C = \sup |c_k|^{1/k}$ where $-f'(z)/f(z) = c_1 + c_2 z + c_3 z^2 + \cdots$. It is shown there that $R \geqq C^{-1}$ and that this result is "best possible" in the sense that there exist functions $f(z)$ for which the above product fails to converge for $|z| > C^{-1}$. (On the other hand, there exist functions whose product-expansion converges beyond this "minimum" circle.) The proof consists in obtaining estimates on the moduli

of the a_k in terms of the moduli of the c_k. While the basic idea of the proof is quite simple, some of the necessary estimates are rather intricate. Whether such products possess any important applications remains an open question.

4. *Articulation of secondary and college mathematics*, by Mr. P. J. Kiernan, The Lawrenceville School.

Cooperating members of the secondary schools and colleges are now attempting to make the union between secondary school and college mathematics smooth for the student. Interest in the problem was aroused by a panel meeting in December, 1949, held under the auspices of the New Jersey Committee on Articulation of Colleges and Secondary Schools. Aside from the curriculum, the major point brought up was that of *reading*. While the secondary schools need to do more in the field of reading, they are handicapped by the meager explanations in current texts. This suggests that the college teacher must continue the training of the student in the intensive type of reading that mathematics requires. A sub-committee is trying to determine an essential curriculum core for secondary school mathematics which is necessary for later success in college work. If the number of items in this core can be kept small, it should stimulate schools in New Jersey to offer not only those but also perhaps some more advanced topics.

C. O. OAKLEY, *Secretary*

OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at the North Dakota Agricultural College in Fargo, North Dakota, on Saturday, October 6, 1951. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors A. G. Hill, P. A. Rognlie and H. D. Colson, Chairman of the Section, presided at the respective sessions.

Thirty-one persons attended the meeting, including the following twenty members of the Association:

H. M. Anderson, Agnes J. Beckstrom, E. J. Camp, H. D. Colson, R. A. Forseth, G. C. Francis, Ruby M. Grimes, K. L. Hankerson, W. H. McBride, N. S. Mendelsohn, Sigurd Mundhjeld, E. O. Nelson, J. C. Peterson, P. A. Rognlie, S. C. Simonson, F. C. Smith, I. W. Smith, R. C. Staley, O. E. Stanaitis, Matilda B. Thompson.

By invitation of the Executive Committee, Professor W. H. McEwen of the University of Manitoba delivered an address at the morning session entitled *Spectral Theory and Differential Eigenvalue Problems*. Abstract of this address follows:

The fundamental notions of Hilbert Space theory were reviewed and the spectrum of a self-adjoint operator H defined. The properties of the point spectrum (of eigenvalues) and the continuous spectrum were noted, and it was shown in some detail how an arbitrary element f in the subspace determined by the eigensolutions can be expanded by means of a limiting process applied to a contour integral of $(H-\lambda I)^{-1}f$. The appropriateness of this formula for the expansion of elements in the whole space was taken for granted, and it was shown how the formula may be used to solve expansion problems associated with the differential equation $y''+(\lambda-q(x))y=0$ in which $q(x)$ is real valued and continuous in an open integral $a<x<b$, the Hilbert Space being the function class $L_2(a, b)$. By way of illustration the solution of the expansion problem for the case $q(x)\equiv 0$ on $-\infty<x<\infty$ was given.

The following seven short papers were presented:

1. *A note on the teaching of complex numbers*, by Professor E. J. Camp, Macalester College.

The purpose of the paper is to show how to "discover" an adequate definition of addition and multiplication of complex numbers regarded as ordered pairs of real numbers. It is required that the new system should have the ordinary real numbers as a subclass; it should form a field; and the system should be closed with respect to the operation of finding roots.

The definition of addition follows immediately from the corresponding definition for vectors. The definition of multiplication is made first for the real complex numbers $(a_1, 0)$ and $(b_1, 0)$. The definition is then generalized in such a way that the field laws are satisfied, and so that the system is closed with respect to the operation of finding roots. These requirements lead to the usual definition of product, *viz.*

$$(a_1, a_2)(b_1, b_2) = (a_1b_1 - a_2b_2, a_2b_1 + a_1b_2)$$

without the necessity of multiplying (a_1+a_2i) by (b_1+b_2i) as if they were two real binomials and then replacing i^2 by -1.

2. *Annuities certain with variable payments*, by Professor F. C. Smith, College of St. Thomas.

In this paper, the author applies the method of inverse differences to the problem of determining the amount and present value of several types of annuities with variable payments.

3. *On Wren's method of finding the highest common factor of two or more polynomials*, by Professor F. J. Arena, North Dakota Agricultural College, presented by Professor W. L. Woodley who was introduced by Professor A. G. Hill.

Dr. Wren has developed a very ingenious method of finding the highest common factor of two or more polynomials and thus the equal roots of two or more algebraic equations. His method possesses several advantages over other known methods: it is very easy to apply and, in the case of algebraic equations, it tells us what the equal roots are. His method is based on some researches which he published in 1937 in the *Bulletin of the American Mathematical Society*, pages 823–834. The purpose of this paper is to explain this method so that it will be available to a larger number of those who have occasion to use it.

4. *Euler's summation formula and convergence of series*, by Professor O. E. Stanaitis, St. Olaf College.

The formula referred to may be used as a criterion in the theory of convergence and divergence of series. It is remarkable that the formula is applicable in cases where the criteria presented in texts fail. The usefulness of the formula has been illustrated by several examples.

5. *Some problems relating to present pension systems*, by Mr. I. W. Smith, 1126 Thirteenth Street North, Fargo, North Dakota.

The work of mathematicians has developed a substantial basis for pension systems, especially in the field of annuities, probability and statistics. Actuaries have made possible many different plans of pension systems which have functioned on a very sound basis.

The three factors in determining a pension system are interest, amount of contribution to the cost and amount of retirement benefit at a given age.

The disturbing elements to earlier-formed pension plans are present low rates of interest on investments of pension funds, the high index of living cost and taxes required under present war budgets. The employee and employer are affected by the latter two factors in support of a satisfactory retirement income.

Pension systems adopted before the present-day conditions require a revision and adjustment to support an adequate amount of retirement benefit.

6. *Use of finite fields in design of experiments*, by Mr. David Gosslee, North Dakota Agricultural College, introduced by the Secretary.

For some large and complicated experimental designs the task of assigning treatments to the experimental units in proper combinations becomes very tedious. The methods due to R. C. Bose and others can greatly reduce the labor. Methods for obtaining treatment combinations for a 3×3 Latin square and a 3^2 factorial design are discussed. The correspondence between the lines and points of a finite geometrical space and certain experimental elements is shown.

7. *An application of a famous inequality*, by Professor N. S. Mendelsohn, University of Manitoba.

Use is made of the classical inequality which states that the arithmetic mean of n positive quantities is greater than or equal to their geometric mean, in order to obtain a very concise proof of the fact that the sequence $(1+1/n)^n$ has a limit as n approaches infinity.

F. C. SMITH, *Secretary*

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The fourteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at Stanford University on Saturday, January 26, 1952. Professor C. D. Olds, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was one hundred and eighteen including the following fifty-one members of the Association:

H. L. Alder, H. M. Bacon, T. J. Bass, Alice K. Bell, D. C. Benson, M. T. Bird, Z. W. Birnbaum, R. N. Bradt, L. P. Burton, P. F. Byrd, H. M. Cochran, Roy Dubisch, Hazel Eggett, D. J. Ewy, Ruth A. Fish, S. A. Francis, C. M. Fulton, L. C. Graue, C. A. Hayes, J. G. Herriot, Marjorie L. Hoffman, Mary T. Huggins, V. F. Ivanoff, Free Jamison, R. M. Lakness, Charles Loewner, J. C. C. McKinsey, R. B. Merkel, E. D. Miller, H. S. Moredock, F. R. Morris, W. H. Myers, C. D. Olds, H. J. Osner, M. F. Pollack, George Polya, Edris P. Rahn, Raphael M. Robinson, E. B. Roessler, Kathryn B. Rolfe, H. L. Royden, Ruth G. Sumner, Mary V. Sunseri, Irving Sussman, A. V. Sylwester, Gabor Szegö, J. L. Ullman, M. J. Vitousek, L. A. Walker, Elizabeth Winstock, A. R. Williams.

The following officers were elected for the coming year: Chairman, Professor Raphael M. Robinson, University of California; Vice-Chairman, Professor Roy Dubisch, Fresno State College; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

The following motion prepared by Professor E. B. Roessler and Professor F. R. Morris was approved:

The Joint Committee on Mathematical Education, Northern and Southern California Sections, Mathematical Association of America, shall have membership rules as follows:

(1) The committee shall be composed of the sectional governors, any governor at large who may live within the territory of the two sections, three additional members from the Northern Section, and three additional members from the Southern Section.

(2) The sectional governor of each section shall appoint one member to the committee in July of each year to serve a term of three years.

(3) The committee as a whole shall elect its own chairman for a period of one year; if for some reason the committee has no chairman, the senior governor shall act as chairman until the committee elects otherwise.

(4) The chairman shall not be eligible to serve more than three consecutive years.

The following resolution proposed by Professor H. S. Moredock was approved:

Whereas it has been found that approximately two-thirds of the people teaching arithmetic in the ninth and tenth grades have neither teaching majors nor minors in mathematics and,

Whereas the number of pupils reaching this level and requiring instruction in arithmetic will certainly increase markedly in the immediate future and,

Whereas an alarming shortage in the number of candidates for the general secondary credential with teaching majors or minors in mathematics even now exists,

Therefore, be it resolved that teacher training institutions be urged to insist upon and to provide opportunity to acquire competence in teaching arithmetic on the part of all candidates seeking general secondary credentials and further that the Joint Committee on Mathematical Education of the Northern and Southern California Sections of the Mathematical Association of America be asked to give early and careful consideration to this problem.

By invitation of the Section, Professor M. Schiffer of the Institute of Advanced Study and Stanford University gave an address entitled *Mathematics and Science* during the morning session.

The following papers were presented:

1. *The volume of the sphere and Archimedes' discovery of the integral calculus*, by Professor George Polya, Stanford University.

In his work on *Method* (which was considered as lost till the beginning of this century) Archimedes derives the area of the parabolic segment, the volume of the sphere, the center of gravity of the hemisphere and about a dozen more results on areas, volumes and centers of gravity. He uses a uniform method to solve this variety of problems, which is remarkable in many respects: the method is heuristic, the idea of equilibrium plays an important role in it, and also the idea of coordinate geometry although, of course, the algebraic notation that makes "analytic geometry" was totally unknown to Archimedes. The speaker proposed to enliven the usual treatment of analytic geometry by incorporating the whole work of Archimedes into it—analytic geometry, as usually treated, is badly in need of some enlivening. Using the example of the volume of the sphere, he demonstrated that the essential idea of Archimedes' great discovery appears undistorted, and even more impressive, if the modern notation of analytic geometry is used.

2. *An essentially undecidable axiom system*, by Professor Raphael M. Robinson, University of California, Berkeley.

The axiom system discussed has the primitive concepts 0, S, $+$, $\cdot$, and consists of seven axioms: If $Sa=Sb$, then $a=b$; $0\neq Sb$; if $a\neq 0$, then $a=Sb$ for some b; $a+0=a$; $a+Sb=S(a+b)$; $a\cdot 0=0$; $a\cdot Sb=a\cdot b+a$. As announced at the International Congress of Mathematicians in 1950, this axiom system is essentially undecidable; that is, there is no general method of deciding what statements follow from this axiom system, and the same is true for any consistent extension. The present talk was limited to discussing some elementary consequences of the axioms, and showing that the formula $0+a=a$ does not follow from the axioms.

3. *A simple proof of the binomial theorem*, by Professor C. M. Fulton, University of California, Davis.

This paper has appeared as a classroom note in the April, 1952 issue of this MONTHLY.

4. *The unexpected in mathematics—a museum of pathology*, by Professor J. G. Herriot, Stanford University.

This paper was devoted to a discussion of several examples from various branches of analysis; the examples were chosen to illustrate various types of unexpected behavior. Included were examples of integration of sequences of functions, convergence of double series, and Cantor's set.

5. *Logarithms are exponents*, by Professor M. T. Bird, San Jose State College.

Charles Davies (*Elements of Algebra* on the basis of M. Bourdon, Revised and Re-written, 1877, p. 310) indicated that logarithms may be expressed as simple continued fractions. The author suggested that such a method might be used by algebra teachers to bridge the gap between the definition of a logarithm and the use of a table of logarithms. The method used to find 1/2, 10/21, 21/44, 52/109, and 73/153 (and their decimal approximations) as successive approximations for the common logarithm of 3. It was noted that the theory of continued fractions provides a measure of accuracy for each approximation.

6. *A generalization of the game of Nim*, by Mr. D. C. Benson, Stanford University.

The games of Nim and Kayles are special cases from an extended class of two-party games. For these games one can define winning and losing positions. Further it is possible to define a commutative and associative multiplication of positions and prime positions. Any position can be written as a product of different prime positions, which is called the factorization of the position. If two positions have the same factorization, then they are equivalent. This concept of equivalence possesses the usual reflexive, symmetric, and transitive properties. There is a simple method of determining the factorization of any position. By means of a table of factorizations for the positions consisting of single counters, one can determine whether or not any given position is winning or losing; and given any losing position it is possible to find a good move, *i.e.* a move which transforms the losing position into a winning position.

7. *On the k-ality theory of rings*, by Professor Irving Sussman, University of San Francisco.

The *k*-ality theory of rings, as developed in a series of papers by A. Foster, was commented upon with relation to its applications to analysis over specialized universal algebras.

The theory consists in supplying the algebra with a convenient number of "coordinate systems" in which to operate by defining over the elements a particular transformation (permutation) group especially chosen for the system. This group then determines a group of mappings within the extensive space of functions of any number of variables, according to the direction:

$$g(x, y, \cdots) \xrightarrow{\rho} g_\rho(x, y, \cdots) = \rho^{-1}[g(\rho(x), \rho(y), \cdots)].$$

A particular example consists of taking the two-group of permutations over any commutative ring R with identity, where $\rho(a) = a^* = 1 - a$, and $a^{**} = a$. Then, for example, the "times" operation yields $a \times b \xrightarrow{\rho} a \times' b = a + b - a \times b$.

If $P(x, y, \cdots, 0, 1, \times', \times, +', +, ^*)$ is a true proposition (*e.g.* an identity) in the "mixed" system, so is the "dual" proposition $P(x^*, y^*, \cdots, 1, 0, \times, \times', +, +', ^*)$.

Exploited over an idempotent ring B, using $y^* = 1 + y$, the system $(B, \times, \times')$ subsumes the Boolean algebra B. An extension to p-rings, embracing multi-valued logics, follows.

Such disciplines as measure theories, geometries, topological groups, probability, seem amenable to treatment by the general theory.

MARJORIE L. HOFFMAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, College of St. Scholastica, Duluth, October 11, 1952.
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October 31, 1952.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Delaware, Newark, November 29, 1952.
ROCKY MOUNTAIN
SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March, 1953.
SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.
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TEXAS
UPPER NEW YORK STATE
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No. 1. *Calculus of Variations* by G. A. Bliss, xiii+189 pages.
No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.
No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
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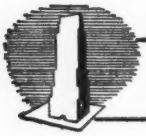

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